A MINIMIZATION PROBLEM ASSOCIATED WITH THE CHERN-SIMONS MODEL WITH DOUBLE VORTEX POINTS ON A TORUS

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Abstract

In this paper, we will study the following minimization problem

$$\inf \left\{ \frac{1}{2} \int_{\Omega} |Du|^2 - 8\pi \ln \int_{\Omega} e^{u+u_0} : u \in H \right\},$$

where \( \{ u \in H^1_{loc}(\mathbb{R}^2) : u \) is doubly periodic in \( \Omega \), and \( \int_{\Omega} u = 0 \}, u_0(x) = -4\pi G(x, p_1) - 4\pi G(x, p_2) \) and \( G(x, p) \) is the Green function of \( -\Delta \) in \( \Omega \) with singularity at \( p \) subject to the periodic boundary condition. We will introduce a quantity \( D(p) \) for \( p \in \Omega \) and prove that if \( D(p) > 0 \) at a maximum point of \( u_0 \), then the above problem has a minimizer.

1. Introduction

Let \( e_1 \) and \( e_2 \) be two linear independent vectors in \( \mathbb{R}^2 \), and let

\( \Omega = \{ x \in \mathbb{R}^2 : x = t_1 e_1 + t_2 e_2, \text{ for some } t_1, t_2 \in [0, 1] \}. \)

Suppose \( p_1 \) and \( p_2 \) are two points in \( \Omega \). Here, \( p_1 \) and \( p_2 \) may coincide. Let

$$u_0(x) = -4\pi G(x, p_1) - 4\pi G(x, p_2), \quad (1.1)$$

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where \( G(x, p_j) \) is the Green function of \(-\Delta\) in \( \Omega \) with singularity at \( p_j \), subject to the periodic boundary condition. That is, \( G(x, p_j) \) satisfies

\[
-\Delta G(x, p_j) = \delta_{p_j} - \frac{1}{|\Omega|},
\]

where \( \delta_{p_j}(x) \) is the Dirac measure at \( p_j \in \Omega \), and \(|\Omega|\) is the measure of \( \Omega \).

Define

\[
I(u) = \frac{1}{2} \int_{\Omega} |Du|^2 - 8\pi \ln \int_{\Omega} e^{u+u_0},
\]

and

\[
H = \{ u \in H^1_{\text{loc}}(\mathbb{R}^2) : u \text{ is doubly periodic with periodic cell } \Omega,
\]

\[
\text{and } \int_{\Omega} u = 0 \}. \tag{1.3}
\]

In this paper, we consider the following minimization problem:

\[
\inf \{ I(u) : u \in H \}. \tag{1.4}
\]

By a Moser-Trudinger type inequality, \( I(u) \) is bounded from below in \( H \). But it is not coercive in \( H \). It turns out that the existence of a minimizer for (1.4) is a very delicate problem.

It is well-known now that the existence of a minimizer of (1.4) is related to the existence of bubbling solution for the following problem:

\[
\begin{cases}
\Delta u + \frac{1}{\varepsilon^2} e^{u+u_0} (1 - e^{u+u_0}) = \frac{8\pi}{|\Omega|}, & \text{in } \Omega, \\
u & \text{is doubly periodic in } \partial \Omega
\end{cases} \tag{1.5}
\]

where and \( \varepsilon \) is the Chern-Simons constant. In fact, Nolasco and Tarantello \([13]\) proved the following result:

**Theorem A.** There is an \( \varepsilon_0 > 0 \), such that for any \( \varepsilon \in (0, \varepsilon_0] \), (1.5) has a solution \( u_\varepsilon \), which can be decomposed to

\[
u_\varepsilon = w_\varepsilon + c_\varepsilon, \quad \int_{\Omega} w_\varepsilon = 0,
\]

for some constant \( c_\varepsilon \) satisfying \( c_\varepsilon \to -\infty \) as \( \varepsilon \to 0 \). And up to a subsequence, one of the following is true
if \( (1.4) \) is achieved, then \( w_\varepsilon \to w \) in \( C^q(\Omega) \) for any \( q \geq 0 \) as \( \varepsilon \to 0 \), and \( w \) is a minimizer of \( (1.4) \);

(ii) if \( (1.4) \) is not achieved, then there exists a \( p_0 \in \Omega \), satisfying \( u_0(p_0) = \max_{x \in \Omega} u_0(x) \) and

\[
\frac{e^{w_\varepsilon + u_0}}{\int_{\Omega} e^{w_\varepsilon + u_0}} \to \delta_{p_0},
\]

in the sense of measure.

Concerning the minimization problem \( (1.4) \), the following result was proved in [5]:

**Theorem B.** Suppose that \( \Omega \) is a rectangle and \( p_1 = p_2 \). Then \( (1.4) \) is not achieved.

More result on this problem can be found in [9].

Note that generally it is very difficult to check whether \( (1.4) \) is achieved or not. The aim of this paper is to give another condition, which will guarantee the existence of a minimizer for \( (1.4) \).

For any \( p \in \Omega \), we define the following quantity

\[
D(p) = \lim_{r \to 0} \int_{\Omega \setminus B_r(p)} \frac{e^{8\pi(\gamma(y,p)-\gamma(p,p))+u_0(y)-u_0(p)} - 1}{|y-p|^4} dy - \int_{\mathbb{R}^2 \setminus \Omega} \frac{1}{|y-p|^2} dy,
\]

where \( \gamma(y,p) \) is the regular part of the Green function \( G(y,p) \). Noting that \( \Delta u_0 + 8\pi \Delta \gamma(y,p) = 0 \), we find that \( D(p) \) is well defined.

The main result of this paper is the following:

**Theorem 1.1.** If there exists a maximum point \( p \) of \( u_0 \) with \( D(p) > 0 \), then \( (1.4) \) is achieved.

The quantity \( D(p) \) was first introduced in [2]. In the last two decades, existence of bubbling solutions have been studied extensively for Chern-Simons model \([1, 6, 7, 10, 11, 12]\) and the related mean field equation \([3, 4, 8]\). When the number of the vortex points is bigger than two, the Chern-Simons model always has bubbling solutions. The case for two vortex points is very delicate. If \( D(p) < 0 \) at a maximum point of \( u_0 \), \( (1.5) \) has a bubbling solution. See [11]. Here, we show that if \( D(p) > 0 \) at a maximum point of \( u_0 \), \( (1.4) \) is achieved. This result complements to the result in [11].
In the rest of this section, let us explain the proof of Theorem 1.1.

In [13], Nolasco and Tarantello constructed a sequence of solution $u_\varepsilon$ for (1.5), satisfying

(i) $u_\varepsilon = w_\varepsilon + c_\varepsilon$ with $\int_\Omega w_\varepsilon = 0$ and

$$e^{c_\varepsilon} = \frac{16\pi \varepsilon^2}{\int_\Omega e^{u_0 + w_\varepsilon} \left(1 + \sqrt{1 - 32\pi \varepsilon^2 \frac{\int_\Omega e^{2(u_0 + w_\varepsilon)}}{\int_\Omega e^{2(u_0 + w_\varepsilon)}}} \right)}.$$  

(1.6)

(ii) $I(w_\varepsilon) \to \inf_{u \in H} I(u)$ as $\varepsilon \to 0$.

This construction shows that (1.4) has special minimization sequence, which is related to the solution of the Chern-Simons model (1.5). Nolasco and Tarantello [13] proved that the sequence $w_\varepsilon$ is either convergent strongly in $H$ to a minimizer of $I(u)$, or blow-up at a maximum point $p$ of the function $u_0$. In the blow-up case, Lemma 4.12 in [13] shows

$$I(w_\varepsilon) \to -8\pi \left(4\pi \gamma(p, p) + u_0(p) + \ln \pi + 1\right),$$

as $\varepsilon \to 0$. Thus, to prove that (1.4) is achieved, we only need to show that

$$\inf_{u \in H} I(u) < -8\pi \left(4\pi \gamma(p, p) + u_0(p) + \ln \pi + 1\right).$$

(1.7)

So, in the next section, we will construct a suitable function $w \in H$, such that

$$I(w) < -8\pi \left(4\pi \gamma(p, p) + u_0(p) + \ln \pi + 1\right),$$

which will imply (1.7).

2. The Proof of the Main Result

Without loss of generality, in this paper, we assume $|\Omega| = 1$.

Let $p \in \Omega$ be a maximum point of $u_0$. Denote

$$V_{p, \mu}(y) = \ln \frac{8\mu^2}{(1 + \mu^2|y - p|^2)^2}, \quad \mu > 0.$$
Then, $u = V_{p,\mu}$ is a solution of
\[ \begin{cases}
-\Delta u = e^u, & \text{in } \mathbb{R}^2; \\
\int_{\mathbb{R}^2} e^u = 8\pi.
\end{cases} \]

Here, we always assume that the constant $\mu > 0$ is large.

In this section, we will construct a function $w_\mu \in H$ such that
\[ I(w_\mu) < -8\pi \left( 4\pi \gamma(p, p) + u_0(p) + \ln \pi + 1 \right). \tag{2.1} \]

Firstly, we define $w_\mu^*$ as follows.
\[ w_\mu^*(y) = \begin{cases}
V_{p,\mu}(y) + 8\pi \left( \gamma(y, p) - \gamma(p, p) \right) \left( 1 - \frac{1}{\theta \mu^2} \right), & y \in B_d(p); \\
V_{0,\mu}(d) + 8\pi \left( G(y, p) - \frac{1}{2\pi} \ln \frac{1}{d} - \gamma(p, p) \right) \left( 1 - \frac{1}{\theta \mu^2} \right), & y \in \Omega \setminus B_d(p),
\end{cases} \tag{2.2} \]
where $\theta > 0$ is a fixed small constant, and the constant $d$ is chosen to make $w_\mu^* \in C^1(\Omega)$. Thus, $d$ satisfies
\[ \mu V'(\mu d) = -\frac{4}{d} \left( 1 - \frac{1}{\theta \mu^2} \right), \]
where $V(y) = V_{0,1}(y)$, which gives
\[ \frac{(\mu d)^2}{1 + (\mu d)^2} = 1 - \frac{1}{\theta \mu^2}. \]

So, we obtain
\[ d^2 = \theta + O\left( \frac{1}{\mu^2} \right). \tag{2.3} \]

Then, we take $L_\mu = -\int_{\Omega} w_\mu^*(y) \, dy$, and
\[ w_\mu(y) = w_\mu^*(y) + L_\mu. \tag{2.4} \]

So $\int_{\Omega} w_\mu(y) \, dy = 0$. We will prove in this section that if $\mu > 0$ is large enough, (2.1) holds. We have
Proposition 2.2. We have

\[ I(w_\mu) = -8\pi(4\pi\gamma(p,p) + \max_{x \in \Omega} u_0(x) + \ln \pi + 1) - \frac{8D(p)}{\mu^2} + O\left(\frac{\ln \mu}{\mu^4}\right). \]

Proof. We have

\[ I(w_\mu) = \frac{1}{2} \int_\Omega |Dw_\mu|^2 - 8\pi \ln \int_\Omega e^{w_\mu + L_\mu + u_0(p) + u_0(y) - u_0(p)} \]
\[ = \frac{1}{2} \int_\Omega |Dw_\mu|^2 - 8\pi L_\mu - 8\pi u_0(p) - 8\pi \ln \int_\Omega e^{w_\mu + u_0(y) - u_0(p)}. \quad (2.5) \]

We estimate \( \frac{\partial I(w_\mu)}{\partial \mu} \) first:

\[ \frac{\partial I(w_\mu)}{\partial \mu} = -\int_\Omega \Delta w_\mu \frac{\partial w_\mu}{\partial \mu} - 8\pi \frac{\partial L_\mu}{\partial \mu} - \frac{8\pi \int_\Omega e^{w_\mu + u_0(y) - u_0(p)} \frac{\partial w_\mu^*}{\partial \mu}}{\int_\Omega e^{w_\mu + u_0(y) - u_0(p)}}. \quad (2.6) \]

Noting that \( \frac{\partial w_\mu}{\partial \mu} = \frac{\partial w_\mu^*}{\partial \mu} + \frac{\partial L_\mu}{\partial \mu} \) and \( \int_\Omega \Delta w_\mu = 0 \), we find

\[ -\int_\Omega \Delta w_\mu \frac{\partial w_\mu}{\partial \mu} = -\int_\Omega \Delta w_\mu \frac{\partial w_\mu^*}{\partial \mu} - \frac{\partial L_\mu}{\partial \mu} \int_\Omega \Delta w_\mu \]
\[ = -\int_\Omega \Delta w_\mu \frac{\partial w_\mu^*}{\partial \mu} \]
\[ = \int_\Omega (1_{B_d(p)} e^{V_{p,\mu}} - 8\pi (1 - \frac{1}{\theta \mu^2})) \frac{\partial w_\mu^*}{\partial \mu}. \quad (2.7) \]

Since \( L_\mu = -\int_\Omega w_\mu^* \), we find

\[ -\int_\Omega \frac{\partial w_\mu^*}{\partial \mu} = \frac{\partial L_\mu}{\partial \mu}. \quad (2.8) \]

So, we obtain from (2.7) and (2.8)

\[ -\int_\Omega \Delta w_\mu \frac{\partial w_\mu}{\partial \mu} - 8\pi \frac{\partial L_\mu}{\partial \mu} = \int_\Omega (1_{B_d(p)} e^{V_{p,\mu}} + \frac{8\pi}{\theta \mu^2}) \frac{\partial w_\mu^*}{\partial \mu}. \quad (2.9) \]

Write

\[ w_\mu^* = V_{p,\mu} + \xi_\mu, \]
where
\[\xi_\mu = \begin{cases} 8\pi (\gamma(y, p) - \gamma(p, p))(1 - \frac{1}{\theta \mu^2}), & \text{in } B_d(p), \\ V_\mu(d) - V_{p,\mu} + 8\pi (G(y, p) - 1) \ln 1 - \gamma(p, p))(1 - \frac{1}{\theta \mu^2}), & \text{in } \Omega \setminus B_d(p). \end{cases}\]

Then
\[\frac{\partial \xi_\mu}{\partial \mu} = \begin{cases} 8\pi (\gamma(y, p) - \gamma(p, p)) \frac{2}{\theta \mu^2}, & \text{in } B_d(p), \\ \frac{\partial}{\partial \mu} (V_\mu(d) - V_{p,\mu}) + 8\pi (G(y, p) - 1) \ln 1 - \gamma(p, p)) \frac{2}{\theta \mu^2}, & \text{in } \Omega \setminus B_d(p). \end{cases}\]

Using the above estimate, we find
\[
\int_\Omega 1_{B_d(p)} e^{V_{p,\mu} \partial \xi_\mu / \partial \mu} = 16\pi \int_{B_d(p)} e^{V_{p,\mu} (\gamma(y, p) - \gamma(p, p))} = O\left(\frac{\ln \mu}{\mu^5}\right).
\]

and
\[
\frac{1}{\theta \mu^2} \int_\Omega \frac{\partial \xi_\mu}{\partial \mu} = \frac{1}{\theta \mu^2} \int_{\Omega \setminus B_d(p)} \frac{\partial}{\partial \mu} (V_\mu(d) - V_{p,\mu}) + O\left(\frac{1}{\mu^5}\right).
\]

Combining (2.9), (2.10), and (2.11), we are led to
\[
- \int_\Omega \Delta w_\mu \frac{\partial w_\mu}{\partial \mu} - 8\pi \frac{\partial L_\mu}{\partial \mu} = \int_\Omega (1_{B_d(p)} e^{V_{p,\mu} + \frac{8\pi}{\theta \mu^2}}) \frac{\partial V_{p,\mu}}{\partial \mu} + O\left(\frac{\ln \mu}{\mu^5}\right),
\]

On the other hand, by (2.3), we have
\[
\int_\Omega (1_{B_d(p)} e^{V_{p,\mu} + \frac{8\pi}{\theta \mu^2}}) \frac{\partial V_{p,\mu}}{\partial \mu} = \int_{B_d(p)} e^{V_{p,\mu} \partial V_{p,\mu} / \partial \mu} + \frac{8\pi}{\theta \mu^2} \left( - \frac{2}{\mu} + \int_\Omega \frac{4}{\mu (1 + \mu^2 |y - p|^2)} \right) - \frac{16\pi}{\theta \mu^3} + O\left(\frac{\ln \mu}{\mu^5}\right).
\]
\[
\begin{align*}
&= \int_{B_d(x)} e^{V_{x,\mu}} \frac{\partial V_{x,\mu}}{\partial \mu} - \frac{16\pi}{\mu^3} + O\left(\frac{\ln \mu}{\mu^5}\right) \\
&= \int_{B_d(x)} e^{V_{x,\mu}} \frac{\partial V_{x,\mu}}{\partial \mu} - \frac{16}{\mu^3} \int_{\mathbb{R}^2 \setminus B_d(x)} \frac{dy}{|y-x|^4} + O\left(\frac{\ln \mu}{\mu^5}\right) \\
&= \int_{B_d(x)} e^{V_{x,\mu}} \frac{\partial V_{x,\mu}}{\partial \mu} + \int_{\mathbb{R}^2 \setminus B_d(x)} e^{V_{x,\mu}} \frac{\partial V_{x,\mu}}{\partial \mu} + O\left(\frac{\ln \mu}{\mu^5}\right) \\
&= \int_{\mathbb{R}^2} e^{V_{x,\mu}} \frac{\partial V_{x,\mu}}{\partial \mu} + O\left(\frac{\ln \mu}{\mu^5}\right) = O\left(\frac{\ln \mu}{\mu^5}\right),
\end{align*}
\]

which, together with (2.12), gives

\[
- \int_{\Omega} \Delta w_{\mu} \frac{\partial w_{\mu}}{\partial \mu} - 8\pi \frac{\partial L_{\mu}}{\partial \mu} = O\left(\frac{\ln \mu}{\mu^5}\right).
\]  

(2.14)

Inserting (2.14) into (2.6), we find

\[
\frac{\partial I(w_{\mu})}{\partial \mu} = -\frac{8\pi}{\mu \theta^3} \int_{\Omega} e^{w_{\mu} + u_0(y) - u_0(p)} \frac{\partial w_{\mu}^*}{\partial \mu} + O\left(\frac{\ln \mu}{\mu^5}\right) + O\left(\frac{1}{\mu^3}\right).
\]  

(2.15)

Let

\[
f(y, p) = 8\pi \left(\gamma(y, p) - \gamma(p, p)\right) + u_0(y) - u_0(p).
\]

(2.16)

It is easy to check that

\[
\int_{\Omega} e^{w_{\mu} + u_0(y) - u_0(p)} \frac{\partial \xi_{\mu}}{\partial \mu} = \frac{16\pi}{\theta^3} \int_{B_d(p)} e^{V_{p,\mu} + f(y, p)} (\gamma(y, p) - \gamma(p, p)) + O\left(\frac{1}{\mu^3}\right)
\]

\[
= O\left(\frac{\ln \mu}{\mu^5}\right),
\]

(2.17)

and

\[
\int_{\Omega} e^{w_{\mu} + u_0(y) - u_0(p)} = 8\pi + O\left(\frac{\ln \mu}{\mu^2}\right).
\]

(2.18)

From (2.15), (2.17) and (2.18), we obtain

\[
\frac{\partial I(w_{\mu})}{\partial \mu} = -\frac{8\pi}{\mu \theta^3} \int_{\Omega} e^{w_{\mu} + u_0(y) - u_0(p)} \frac{\partial V_{p,\mu}}{\partial \mu} + O\left(\frac{\ln \mu}{\mu^5}\right).
\]

(2.19)

Let

\[
f_{\mu}(y, p) = 8\pi \left(\gamma(y, p) - \gamma(p, p)\right) \left(1 - \frac{1}{\theta^3}\right) + u_0(y) - u_0(p).
\]

(2.20)
Then
\[
\begin{align*}
\int_{\Omega} e^{w^*_\mu + u_0(y) - u_0(p)} \frac{\partial V_{p,\mu}}{\partial \mu} & = \int_{B_d(p)} e^{V_{p,\mu} + f_\mu(y,p)} \frac{\partial V_{p,\mu}}{\partial \mu} + \int_{\Omega \setminus B_d(p)} e^{w^*_\mu + u_0(y) - u_0(p)} \frac{\partial V_{p,\mu}}{\partial \mu} \\
& = \int_{B_d(p)} e^{V_{p,\mu} + f_\mu(y,p)} \frac{\partial V_{p,\mu}}{\partial \mu} + \frac{8\mu^2}{(1 + \mu^2d^2)^2} \int_{\Omega \setminus B_d(p)} \frac{d^4(1 - \frac{1}{\mu^2})}{\frac{1}{\mu^2} |y - p|^4} e^{f_\mu(y,p)} \frac{\partial V_{p,\mu}}{\partial \mu} + O\left(\frac{1}{\mu^3}\right) \\
& = \int_{B_d(p)} e^{V_{p,\mu} + f_\mu(y,p)} \frac{\partial V_{p,\mu}}{\partial \mu} + \frac{8}{\mu^2} \int_{\Omega \setminus B_d(p)} \frac{1}{|y - p|^4} e^{f_\mu(y,p)} \frac{\partial V_{p,\mu}}{\partial \mu} + O\left(\frac{1}{\mu^3}\right) \\
& = \int_{\Omega} e^{V_{p,\mu}} \frac{\partial V_{p,\mu}}{\partial \mu} + \int_{\Omega} e^{V_{p,\mu}} (e^{f_\mu(y,p) - 1}) \frac{\partial V_{p,\mu}}{\partial \mu} + O\left(\frac{1}{\mu^3}\right). \quad (2.21)
\end{align*}
\]

On the other hand,
\[
\begin{align*}
\int_{\Omega} e^{V_{p,\mu}} \frac{\partial V_{p,\mu}}{\partial \mu} & = \frac{\partial}{\partial \mu} \int_{\Omega} e^{V_{p,\mu}} = -\frac{\partial}{\partial \mu} \int_{\mathbb{R}^2 \setminus \Omega} e^{V_{p,\mu}} = \frac{16}{\mu^3} \int_{\mathbb{R}^2 \setminus \Omega} \frac{1}{|y - p|^4} dy + O\left(\frac{1}{\mu^3}\right). \quad (2.22)
\end{align*}
\]

Noting that
\[
e^{f_\mu(y,p) - 1} = e^{f(y,p) - 1} + O\left(\frac{1}{\mu^2}\right),
\]

we find
\[
\begin{align*}
\int_{\Omega} e^{V_{p,\mu}} (e^{f_\mu(y,p) - 1}) \frac{\partial V_{p,\mu}}{\partial \mu} & = \int_{\Omega} e^{V_{p,\mu}} (e^{f(y,p) - 1}) \frac{\partial V_{p,\mu}}{\partial \mu} + O\left(\frac{1}{\mu^3}\right). \quad (2.23)
\end{align*}
\]

Combining (2.21), (2.22) and (2.23), we are led to
\[
\begin{align*}
\int_{\Omega} e^{w^*_\mu + u_0(y) - u_0(p)} \frac{\partial V_{p,\mu}}{\partial \mu} & = \frac{16}{\mu^3} \int_{\mathbb{R}^2 \setminus \Omega} \frac{1}{|y - p|^4} dy + \int_{\Omega} e^{V_{p,\mu}} (e^{f(y,p) - 1}) \frac{\partial V_{p,\mu}}{\partial \mu} + O\left(\frac{\ln \mu}{\mu^3}\right) \\
& = \frac{16}{\mu^3} \int_{\mathbb{R}^2 \setminus \Omega} \frac{1}{|y - p|^4} dy - \int_{\Omega} \frac{16}{\mu^3 |y - p|^4} (e^{f(y,p) - 1}) + O\left(\frac{\ln \mu}{\mu^3}\right) \\
& = -\frac{16D(p)}{\mu^3} + O\left(\frac{\ln \mu}{\mu^3}\right). \quad (2.24)
\end{align*}
\]
From (2.24) and (2.18), we obtain
\[
-8\pi \int_{\Omega} e^{w^*_t+u_0(y)-u_0(p)} \frac{\partial w^*_t}{\partial \mu} + 16D(p) \frac{\ln \mu}{\mu^3} = O\left(\frac{\ln \mu}{\mu^3}\right).
\] (2.25)

Combining (2.19) and (2.25), we are led to
\[
\frac{\partial I(w_\mu)}{\partial \mu} = 16D(p) \frac{\ln \mu}{\mu^3} + O\left(\frac{\ln \mu}{\mu^5}\right).
\] (2.26)

We are now ready to estimate \( I(w_\mu) \). From (2.26), we find
\[
\lim_{t \to +\infty} I(w_t) - I(w_\mu) = \frac{8D(p)}{\mu^2} + O\left(\frac{\ln \mu}{\mu^4}\right),
\] (2.27)
which gives
\[
I(w_\mu) = \lim_{t \to +\infty} I(w_t) - \frac{8D(p)}{\mu^2} + O\left(\frac{\ln \mu}{\mu^4}\right).
\] (2.28)

Finally, we estimate \( \lim_{t \to +\infty} I(w_t) \). By (2.5),
\[
I(w_t) = -\frac{1}{2} \int_{\Omega} w_t \Delta w_t - 8\pi \ln \int_{\Omega} e^{w^*_t+u_0(y)-u_0(p)} - 8\pi L_t - 8\pi u_0(p) \] (2.29)

But from \( \int_{\Omega} w_t = 0 \),
\[
-\frac{1}{2} \int_{\Omega} w_t \Delta w_t = \frac{1}{2} \int_{B_d(p)} e^{V_{p,t}}w_t + 4\pi(1 - \frac{1}{\theta t^2}) \int_{\Omega} w_t = \frac{1}{2} \int_{B_d(p)} e^{V_{p,t}}w_t.
\] (2.30)

It is easy to check
\[
\int_{\Omega} e^{w^*_t+u_0(y)-u_0(p)} \to 8\pi, \quad \text{as } t \to +\infty,
\] (2.31)
and
\[
\frac{1}{2} \int_{B_d(p)} e^{V_{p,t}}w_t - 8\pi L_t
\]
\[
= \frac{1}{2} \left( \ln(8t^2) + L_t \right) \int_{B_d(p)} e^{V_{p,t}} - 8\pi L_t + \frac{1}{2} \int_{B_d(p)} e^{V_{p,t}} \ln \frac{1}{(1 + t^2|y-p|^2)^2}
\]
\[
+ \frac{1}{2} \int_{B_d(p)} e^{V_{p,t}}8\pi (\gamma(y,p) - \gamma(p,p))(1 - \frac{1}{\theta t^2})
\]
\[
\begin{align*}
&= \frac{1}{2} (\ln(8t^2) + L_t) \left(8\pi + O\left(\frac{1}{t^2}\right)\right) - 8\pi L_t + \int_{\mathbb{R}^2} \frac{4}{(1 + |y|^2)^2} \ln \frac{1}{(1 + |y|^2)^2} + o(1) \\
&= \frac{1}{2} (\ln(8t^2) - L_t) 8\pi - 8\pi + o(1) + O\left(\frac{L_t}{t^2}\right),
\end{align*}
\]

(2.32)

since
\[
\begin{align*}
\int_{\mathbb{R}^2} \frac{4}{(1 + |y|^2)^2} \ln \frac{1}{(1 + |y|^2)^2} &= 8\pi \int_0^{+\infty} \frac{1}{(1 + t)^2} \ln \frac{1}{1 + t} dt \\
&= 8\pi \int_0^{+\infty} \frac{1}{(1 + t)^2} dt = 8\pi.
\end{align*}
\]

Moreover, from \( \int_{\Omega} G(y, p) \, dy = 0 \), we get
\[
\int_{\Omega} \gamma(y, p) \, dy = -\frac{1}{2\pi} \int_{\Omega} \ln \frac{1}{|y - p|} \, dy.
\]

As a result,
\[
-L_t = \int_{\Omega} w_t^* = \int_{B_d(p)} (V_{p,t} - 4(1 - \frac{1}{\theta t^2}) \ln \frac{1}{|y - p|}) \, dy \\
+ \int_{\Omega \setminus B_d(p)} (V_t(d) - 4(1 - \frac{1}{\theta t^2}) \ln \frac{1}{d}) \, dy - \gamma(p, p) 8\pi (1 - \frac{1}{\theta t^2}) \\
= \pi d^2 \ln(8t^{-2}) + \int_{B_d(p)} \ln \frac{|y - p|^4}{(\frac{1}{t^2} + |y - p|^2)^2} \, dy \\
+ (1 - \pi d^2) \left(\ln \frac{8}{t^2} + \ln \frac{d^4}{(d^2 + \frac{1}{\pi t^2})^2}\right) - 8\pi \gamma(p, p) + o(1) \\
= \ln(8t^{-2}) - 8\pi \gamma(p, p) + o(1). \tag{2.33}
\]

From (2.32) and (2.33), we obtain
\[
\begin{align*}
\frac{1}{2} \int_{B_d(p)} e^{V_{t,t}} w_t - 8\pi L_t \\
= \frac{1}{2} \left(\ln(8t^2) + \ln(8t^{-2}) - 8\pi \gamma(p, p) + o(1)\right) 8\pi - 8\pi + o(1) + O\left(\frac{\ln t}{t^2}\right) \\
= 8\pi (\ln 8 - 4\pi \gamma(p, p) - 1) + o(1). \tag{2.34}
\end{align*}
\]

Combining (2.29), (2.30), (2.31) and (2.34), we obtain
\[
I(w_t) = 8\pi (\ln 8 - 4\pi \gamma(p, p) - 1) - 8\pi u_0(p) - 8\pi \ln(8\pi) + o(1) \\
= -8\pi (4\pi \gamma(p, p) + u_0(p) + \ln \pi + 1) + o(1),
\]
which gives
\[
\lim_{t \to +\infty} I(w_t) = -8\pi \left(4\pi \gamma(p, p) + u_0(p) + \ln \pi + 1\right).
\]

**Proof of Theorem 1.1**. If \(D(p) > 0\), it follows from Proposition 2.2,
\[
\inf_{u \in H} I(u) \leq I(w_\mu) < -8\pi \left(4\pi \gamma(p, p) + u_0(p) + \ln \pi + 1\right),
\]
if \(\mu > 0\) is large enough. Thus the minimizing sequence constructed in [13] converges strongly in \(H\) and (1.4) is achieved.

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**References**


