

LEVEL 1 QUENCHED LARGE DEVIATION PRINCIPLE FOR RANDOM WALK IN DYNAMIC RANDOM ENVIRONMENT

DAVID CAMPOS^{1,a}, ALEXANDER DREWITZ^{2,b},
ALEJANDRO F. RAMÍREZ^{3,c}, FIRAS RASSOUL-AGHA^{4,d},
AND TIMO SEPPÄLÄINEN^{5,e}

¹Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Vicuña Mackenna 4860, Macul, Santiago, Chile.

^aE-mail: jdcampo1@mat.puc.cl

²Departement Mathematik, ETH Zürich, Rämistrasse 101, CH-8092 Zürich, Switzerland.

^bE-mail: alexander.drewitz@math.ethz.ch

³Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Vicuña Mackenna 4860, Macul, Santiago, Chile.

^cE-mail: aramirez@mat.puc.cl

⁴Department of Mathematics, University of Utah, 155 South 1400 East, Salt Lake City, UT 84109, USA.

^dE-mail: fras@math.utah.edu

⁵Department of Mathematics, University of Wisconsin-Madison, 419 Van Vleck Hall, Madison, WI 53706, USA.

^eE-mail: seppalai@math.wisc.edu

Abstract

Consider a random walk in a time-dependent random environment on the lattice \mathbb{Z}^d . Recently, Rassoul-Agha, Seppäläinen and Yilmaz [13] proved a general large deviation principle under mild ergodicity assumptions on the random environment for such a random walk, establishing first level 2 and 3 large deviation principles. Here we present two alternative short proofs of the level 1 large deviations under mild ergodicity assumptions on the environment: one for the continuous time case and another one for the discrete time case. Both proofs provide the existence, continuity and convexity of the rate function. Our methods are based on the use of the sub-additive ergodic theorem as presented by Varadhan in [22].

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1. Introduction

We consider uniformly elliptic random walks in time-space random environment both in continuous and discrete time. We present two alternative short proofs of the level 1 quenched large deviation principle under mild conditions on the environment, based on the use of the sub-additive ergodic theorem as presented by Varadhan in [22]. Previously, in the discrete time case, Rassoul-Agha, Seppäläinen and Yilmaz [13], proved a level 2 and 3 large deviation principle, from which the level 1 principle can be derived via contraction.

Let $\kappa_2 > \kappa_1 > 0$. Denote by $G := \{e_1, e_{-1}, \dots, e_d, e_{-d}\}$ the set of unit vectors in \mathbb{Z}^d . Define $\mathcal{Q} := \{v = \{v(e) : e \in G\} : \kappa_1 \leq \inf_{e \in G} v(e) \leq \sup_{e \in G} v(e) \leq \kappa_2\}$. Consider a continuous time Markov process $\omega := \{\omega_t : t \geq 0\}$ with state space $\Omega_c := \mathcal{Q}^{\mathbb{Z}^d}$, so that $\omega_t := \{\omega_t(x) : x \in \mathbb{Z}^d\}$ with $\omega_t(x) := \{\omega_t(x, e) : e \in G\} \in \mathcal{Q}$. We call ω the *continuous time environmental process*. We assume that for each initial condition ω_0 , the process ω defines a probability measure $Q_{\omega_0}^c$ on the Skorokhod space $D([0, \infty); \Omega_c)$. Let μ be an invariant measure for the environmental process ω so that for every bounded continuous function $f : \Omega_c \rightarrow \mathbb{R}$ and $t \geq 0$ we have that

$$\int f(\omega_t) d\mu = \int f(\omega_0) d\mu.$$

Assume that μ is also invariant under the action of space-translations. Furthermore, we define $Q_\mu^c := \int Q_\omega^c d\mu$, where with a slight abuse of notation here $\omega \in \Omega_c$. For a given trajectory $\omega \in D([0, \infty); \Omega_c)$ consider the process $\{X_t : t \geq 0\}$ defined by the generator

$$L_s f(x) := \sum_{e \in G} \omega_s(x, e) (f(x + e) - f(x)),$$

where $s \geq 0$. We call this process a *continuous time random walk in a uniformly elliptic time-dependent random environment* and denote for each $x \in \mathbb{Z}^d$ by $P_{x, \omega}^c$ the law on $D([0, \infty); \mathbb{Z}^d)$ of this random walk with initial condition $X_0 = x$. We call $P_{x, \omega}^c$ the *quenched law* starting from x of the random walk.

For $x \in \mathbb{R}^d$, $|x|_2$, $|x|_1$ and $|x|_\infty$ denote respectively, their Euclidean, l_1 and l_∞ -norm. Also, for $r > 0$, we define $B_r(x) := \{y \in \mathbb{Z}^d : |y - x|_2 \leq r\}$.

Furthermore, given any topological space T , we will denote by $\mathcal{B}(T)$ the corresponding Borel sets.

We will also consider a discrete version of this model which we define as follows. Let $\kappa > 0$ and $R \subset \mathbb{Z}^d$ finite. Define $\mathcal{P} := \{v = \{v(e) : e \in R\} : \inf_{e \in R} v(e) \geq \kappa, \sum_{e \in R} v(e) = 1\}$. Consider a discrete time Markov process $\omega := \{\omega_n : n \geq 0\}$ with state space $\Omega_d := \mathcal{P}^{\mathbb{Z}^d}$, so that $\omega_n := \{\omega_n(x) : x \in \mathbb{Z}^d\}$ with $\omega_n(x) := \{\omega_n(x, e) : e \in R\} \in \mathcal{P}$. We call ω the *discrete time environmental process*. Let us denote by Q_ω^d the corresponding law of the process defined on the space $\Omega_d^{\mathbb{N}}$. Let μ be an invariant measure for the environmental process ω so that for every bounded continuous function $f : \Omega_d \rightarrow \mathbb{R}$ and $n \geq 0$ we have that

$$\int f(\omega_n) d\mu = \int f(\omega_0) d\mu.$$

Assume that μ is also invariant under the action of space-translations. Furthermore, we define $Q_\mu^d := \int Q_\omega^d d\mu$. Given $\omega \in \Omega_d$ and $x \in \mathbb{Z}^d$, consider now the discrete time random walk $\{X_n : n \geq 0\}$ with a law $P_{x,\omega}^d$ on $(\mathbb{Z}^d)^{\mathbb{N}}$ defined through $P_{x,\omega}^d(X_0 = x) = 1$ and the transition probabilities

$$P_{x,\omega}^d(X_{n+1} = x + e | X_n = x) = \omega_n(x, e),$$

for $n \geq 0$ and $e \in R$. We call this process a *discrete time random walk in a uniformly elliptic time-space random environment with jump range R* and call $P_{x,\omega}^d$ the *quenched law* of the discrete time random walk starting from x . We will say that R *corresponds to the nearest neighbor case* if $R = \{e \in \mathbb{Z}^d : |e|_1 = 1\}$. We say that a subset $A \subset \mathbb{Z}^d$ is *convex* if there exists a convex subset $V \subset \mathbb{R}^d$ such that $A = V \cap \mathbb{Z}^d$, while we say that A is *symmetric* if $A = -A$. Throughout, we will assume that the jump range is R is finite, convex and symmetric or that it corresponds to the nearest neighbor case.

Throughout we will make the following ergodicity assumption. Note that we do not demand the environment to be necessarily ergodic under time shifts.

Assumption (EC). Consider the continuous time environmental process ω . For each $s > 0$ and $x \in \mathbb{Z}^d$ define the transformation $T_{s,x} : D([0, \infty); \Omega_c) \rightarrow D([0, \infty); \Omega_c)$ by $(T_{s,x}\omega)_t(y) := \omega_{t+s}(y + x)$. We say that the environmental

process ω satisfies *assumption (EC)* if $\{T_{s,x} : s > 0, x \in \mathbb{Z}^d\}$ is an ergodic family of transformations acting on the space $(D([0, \infty); \Omega_c), \mathcal{B}(D([0, \infty); \Omega_c)), Q_\mu^c)$. In other words, the latter means that whenever $A \in \mathcal{B}(D([0, \infty); \Omega_c))$ is such that $T_{s,x}^{-1}A = A$ for every $s > 0$ and $x \in \mathbb{Z}^d$, then $Q_\mu^c(A)$ is 0 or 1.

Assumption (ED). Consider the discrete time environmental process ω . For $x \in \mathbb{Z}^d$ define the transformation $T_{1,x} : D([0, \infty); \Omega_d) \rightarrow D([0, \infty); \Omega_d)$ by $(T_{1,x}\omega)_n(y) := \omega_{n+1}(y+x)$. We say that the environmental process ω satisfies *assumption (ED)* if $\{T_{1,x} : x \in R\}$ is an ergodic family of transformations acting on the space $(\Omega_d^{\mathbb{N}}, \mathcal{B}(\Omega_d^{\mathbb{N}}), Q_\mu^d)$. In other words, whenever $A \in \mathcal{B}(\Omega_d^{\mathbb{N}})$ is such that $T_{1,x}^{-1}A = A$ for every $x \in R$, then $Q_\mu^d(A)$ is 0 or 1.

It is straightforward to check that assumption (ED) is equivalent to asking that whenever $A \in \mathcal{B}(\Omega_d^{\mathbb{N}})$ is such that $A = T_{n,x}^{-1}A$ for every $x \in R$ and $n \in \mathbb{N}$ then $Q_\mu^d(A)$ is 0 or 1.

In this paper we present a level 1 quenched large deviation principle for both the continuous and the discrete time random walk in time-space random environment. It should be noted that the discrete time version of our result can be derived via a contraction principle from results that have been obtained in Rassoul-Agha, Seppäläinen and Yilmaz [13] establishing level 2 and 3 large deviations, for discrete time random walks on time-space random environments and potentials. There, the authors also derive variational expressions for the rate functions. Nevertheless, the proofs we present here of both Theorem 1.1 and 1.2, are short and direct.

Theorem 1.1. *Consider a continuous time random walk $\{X_t : t \geq 0\}$ in a uniformly elliptic time-dependent environment ω satisfying assumption (EC). Then, there exists a convex continuous rate function $I_c(x) : \mathbb{R}^d \rightarrow [0, \infty)$ such that the following are satisfied.*

(i) *For every open set $G \subset \mathbb{R}^d$ we have that Q_μ^c -a.s.*

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \log P_{0,\omega}^c \left(\frac{X_t}{t} \in G \right) \geq - \inf_{x \in G} I_c(x).$$

(ii) *For every closed set $C \subset \mathbb{R}^d$ we have that Q_μ^c -a.s.*

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log P_{0,\omega}^c \left(\frac{X_t}{t} \in C \right) \leq - \inf_{x \in C} I_c(x).$$

To state the discrete time version of Theorem 1.1, we need to introduce some notation. Let $R_0 := \{0\} \subset \mathbb{Z}^d$, $R_1 := R$ and for $n \geq 1$ define

$$R_{n+1} := \{y \in \mathbb{Z}^d : y = x + e \text{ for some } x \in R_n \text{ and } e \in R\},$$

and $U_n := R_n/n$. Note that R_n is the set of sites that a random walk with jump range R visits with positive probability at time n . We then define U as the set of limit points of the sequence of sets $\{U_n : n \geq 1\}$, so that

$$U := \{x \in \mathbb{R}^d : x = \lim_{n \rightarrow \infty} x_n \text{ for some sequence } x_n \in U_n\}. \quad (1.1)$$

Theorem 1.2. *Consider a discrete time random walk $\{X_n : n \geq 0\}$ in a uniformly elliptic time-dependent environment ω satisfying assumption (ED) with jump range R . Assume that either (i) R is finite, convex, symmetric and there is a neighborhood of 0 which belongs to the convex hull of R ; (ii) or that R corresponds to the nearest neighbor case. Consider U defined in (1.1). Then U equals the convex hull of R and there exists a convex rate function $I_d(x) : \mathbb{R}^d \rightarrow [0, \infty]$ such that $I_d(x) \leq |\log \kappa|$ for $x \in U$, $I_d(x) = \infty$ for $x \notin U$, I is continuous for every $x \in U^\circ$ and the following are satisfied.*

(i) *For every open set $G \subset \mathbb{R}^d$ we have that Q_μ^d -a.s.*

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log P_{0,\omega}^d \left(\frac{X_n}{n} \in G \right) \geq - \inf_{x \in G} I_d(x).$$

(ii) *For every closed set $C \subset \mathbb{R}^d$ we have that Q_μ^d -a.s.*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log P_{0,\omega}^d \left(\frac{X_n}{n} \in C \right) \leq - \inf_{x \in C} I_d(x).$$

Both quenched and annealed large deviations for discrete time random walks on random environments which do not depend on time, have been thoroughly studied in the case in which $d = 1$ (see the reviews of Sznitman [21] and Zeitouni [26] for both the one-dimensional and multi-dimensional cases). The first quenched multidimensional result was obtained by Zerner in [27] under the so called plain nestling condition, concerning the law of the support of the quenched drift (see also [26] and [21]). In [22], Varadhan established both a general quenched and annealed large deviation principle for discrete time random walks in static random environments via the use

of the subadditive ergodic theorem. In the quenched case, he assumed uniform ellipticity and the ergodicity assumption (ED). Subsequently, in his Ph.D. thesis [17], Rosenbluth extended the quenched result of Varadhan under a condition weaker than uniform ellipticity, along with a variational formula for the rate function (see also Yilmaz [23, 24, 25]). The method of Varadhan based on the subadditive ergodic theorem and of Rosenbluth [17], Yilmaz [23] and Rassoul-Agha, Sepäläinen, Yilmaz [13], are closely related to the use of the subadditive ergodic theorem in the context of non-linear stochastic homogenization (see for example the paper of dal Maso, Modica [12]). Closer and more recent examples of stochastic homogenization for the Hamilton-Jacobi-Bellman equation with static Hamiltonians via the subadditive ergodic theorem are the work of Rezakhanlou and Tarver [15] and of Souganidis [19] and in the context of the totally asymmetric simple K -exclusion processes and growth processes the works of Seppäläinen in [18] and Rezakhanlou in [14]. Stochastic homogenization for the Hamilton-Jacobi-Bellman equation with respect to time-space shifts was treated by Kosygina and Varadhan in [11] using change of measure techniques giving variational expressions for the effective Hamiltonian.

A particular case of Theorem 1.1 is the case of a random walk which has a drift in a given direction on occupied sites and in another given direction on unoccupied sites, where the environment is generated by an attractive spin-flip particle system or a simple exclusion process (see Avena, den Hollander and Redig [1] for the case of a one-dimensional attractive spin-flip dynamics, and also [2, 3, 10]). This case is also included in the results presented in [13]. Another particular case of Theorem 1.1 is a continuous time random walk in a static random environment with a law which is ergodic under spatial translations: two of these cases are the Bouchaud trap random walk with bounded jump rates (see for example [4]) and the continuous time random conductances model (see for example [5]). Our proof would also apply to the polymer measure defined by a continuous time random walk in time-dependent random environment and bounded random potential (see [13]). Note that Theorem 1.2 does include the classical nearest neighbor case (a nearest neighbor case example is the random walk on a time-space i.i.d. environment studied by Yilmaz [24]).

Our proofs are obtained by directly establishing the level 1 large deviation principle and is based on the sub-additive ergodic theorem as used by

Varadhan in [22]. Let us note, that in [22], Varadhan applies sub-additivity directly to the logarithm of a smoothed up version of the inverse of the transition probabilities of the random walk, as opposed to the earlier approach of Zerner [27] (see also Sznitman [20]), where sub-additivity is applied to a generalized Laplace transform of the hitting times of sites of the random walk forcing to assume the so called nestling property on the random walk. While our methods do not give any explicit information about the rate function, besides its convexity and continuity, the proofs are short and simple.

We do not know how to define a smoothed up version of the transition probabilities as is done by Varadhan in [22]. We therefore have to prove directly an equicontinuity estimate for the transition probabilities of the random walk, which is the main difficulty in the proofs of Theorems 1.1 and 1.2. In the case of Theorem 1.1 we follow the method presented in [7]: we first express the transition probabilities of the walk in terms of those of a simple symmetric random walk through a Radon-Nykodym derivative, then through the use of Chapman-Kolmogorov equation we rely on standard large deviation estimates for the continuous time simple symmetric random walk.

In Section 2 we present the proof of Theorem 1.1 using the methods developed in [7]. In Section 3 we continue with the proof of Theorem 1.2 in the case in which the jump range of the walk R is convex, symmetric and a neighborhood of 0 is contained in its convex hull. In Section 4 we prove Theorem 1.2 for the discrete time nearest neighbor case. Throughout the rest of the paper we will use the notations c, C, C', C'' to refer to different positive constants.

2. Proof of Theorem 1.1

For each $s \geq 0$, let $\theta_s : D([0, \infty); \Omega_c) \rightarrow D([0, \infty); \Omega_c)$ denote the canonical time shift. As in [7], we first define for each $0 \leq s < t$ and $x, y \in \mathbb{Z}^d$ the quantities

$$e(s, t, x, y) := P_{x, \theta_s \omega}^c (X_{t-s} = y),$$

and

$$a_c(s, t, x, y) := -\log e(s, t, x, y),$$

where the subscript c in a_c is introduced to distinguish this quantity from the corresponding discrete time one. Note that these functions still depend on the realization of ω . We call $a_c(s, t, x, y)$ the point to point passage function from x to y between times s and t . Due to the fact that we are considering a continuous time random walk, here we do not need to smooth out the point to point passage functions (see [22]). Nevertheless, there is an equicontinuity issue that should be resolved. Theorem 1.1 will follow directly from the following shape theorem. A version of this shape theorem for a random walk in random potential has been established as Theorem 4.1 in [7] (see also Theorem 2.5 of Chapter 5 of Sznitman [20]).

Theorem 2.1 (Shape theorem). *There exists a deterministic convex function $I_c : \mathbb{R}^d \rightarrow [0, \infty)$ such that $Q_\mu^c - a.s.$, for any compact set $K \subset \mathbb{R}^d$*

$$\lim_{t \rightarrow \infty} \sup_{y \in tK \cap \mathbb{Z}^d} \left| t^{-1} a_c(0, t, 0, y) - I_c \left(\frac{y}{t} \right) \right| = 0. \quad (2.1)$$

Furthermore, for any $M > 0$, we can find a compact $K \subset \mathbb{R}^d$ such that $Q_\mu^c - a.s.$

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log P_{0, \omega}^c \left(\frac{X_t}{t} \notin K \right) \leq -M. \quad (2.2)$$

Let us first see how to derive Theorem 1.1 from Theorem 2.1. We will first prove the upper bound of part (ii) of Theorem 1.1. By (2.2) of Theorem 2.1, we know that we can choose a compact set $K \subset \mathbb{R}^d$ such that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log P_{0, \omega}^c \left(\frac{X_t}{t} \notin K \right) < - \inf_{x \in C} I_c(x),$$

where C is a closed set. It is therefore enough to prove that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log P_{0, \omega}^c \left(\frac{X_t}{t} \in C \cap K \right) \leq - \inf_{x \in C} I_c(x).$$

Now,

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{t} \log P_{0, \omega}^c \left(\frac{X_t}{t} \in C \cap K \right) \\ & \leq \limsup_{t \rightarrow \infty} \frac{1}{t} \sup_{y \in (tC \cap tK) \cap \mathbb{Z}^d} \log P_{0, \omega}^c (X_t = y) \end{aligned}$$

$$= \limsup_{t \rightarrow \infty} \frac{1}{t} \log P_{0,\omega}^c (X_t = y_t),$$

where $y_t \in (tC \cap tK) \cap \mathbb{Z}^d$, is a point that maximizes $P_{0,\omega}^c(X_t = \cdot)$. Now, by compactness, there is a subsequence $t_n \rightarrow \infty$ such that

$$\lim_{n \rightarrow \infty} \frac{y_{t_n}}{t_n} =: x^* \in C \cap K,$$

and $\limsup_{t \rightarrow \infty} \frac{1}{t} \log P_{0,\omega}^c (X_t = y_t) = \limsup_{n \rightarrow \infty} \frac{1}{t_n} \log P_{0,\omega}^c (X_{t_n} = y_{t_n})$.

Thus, by the continuity of I_c and by (2.1) we see that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log P_{0,\omega}^c \left(\frac{X_t}{t} \in C \cap K \right) \leq -I_c(x^*) \leq -\inf_{x \in C} I_c(x).$$

To prove the lower bound, part (i) of Theorem 1.1, note that by (2.1) we have that

$$\begin{aligned} \liminf_{t \rightarrow \infty} \frac{1}{t} \log P_{0,\omega}^c \left(\frac{X_t}{t} \in G \right) &\geq \liminf_{t \rightarrow \infty} \frac{1}{t} \sup_{y \in (tG) \cap \mathbb{Z}^d} \log P_{0,\omega}^c (X_t = y) \\ &\geq -\inf_{x \in G} I_c(x). \end{aligned}$$

Let us now continue with the proof of Theorem 2.1. Display (2.2) of Theorem 2.1 follows from standard large deviation estimates for the process $\{N_t : t \geq 0\}$, where N_t is the total number of jumps up to time t of the random walk $\{X_t : t \geq 0\}$, which can be coupled with a Poisson process of parameter $2d\kappa_2$. To prove the first statement (2.1) of Theorem 2.1 we first observe that for every $0 \leq t_1 < t_2 < t_3$ and $x_1, x_2, x_3 \in \mathbb{Z}^d$ one has that Q_μ^c -a.s.

$$a_c(t_1, t_3, x_1, x_3) \leq a_c(t_1, t_2, x_1, x_2) + a_c(t_2, t_3, x_2, x_3). \quad (2.3)$$

We will also need to obtain bounds on the point to point passage functions which will be eventually used to prove some crucial equicontinuity estimates. To prove these bounds, we first state Lemma 4.2 of [7], which is a large deviation estimate for the simple symmetric random walk.

Lemma 2.1. *Let X be a simple symmetric random walk on \mathbb{Z}^d with jump rate κ and starting point $X(0) = 0$. For each $x \in \mathbb{Z}^d$ and $t > 0$ let $p(t, 0, x)$ be the probability that this random walk is at position x at time t starting*

from 0. Then for every $t > 0$ and $x \in \mathbb{Z}^d$, we have

$$p(t, 0, x) = \frac{e^{-J(\frac{x}{t})t}}{(2\pi t)^{\frac{d}{2}} \prod_{i=1}^d \left(\frac{x_i^2}{t^2} + \frac{\kappa^2}{d^2}\right)^{1/4}} (1 + o(1)), \quad (2.4)$$

where

$$J(x) := \sum_{i=1}^d \frac{\kappa}{d} j\left(\frac{dx_i}{\kappa}\right) \quad \text{with} \quad j(y) := y \sinh^{-1} y - \sqrt{y^2 + 1} + 1,$$

and the error term $o(1)$ tends to zero as $t \rightarrow \infty$ uniformly in $x \in tK \cap \mathbb{Z}^d$, for any compact $K \subset \mathbb{R}^d$. Furthermore the function j is increasing with $|y|$ and $j \geq 0$.

We will need the following estimates for the transition probabilities.

Lemma 2.2. *Consider the transition probabilities of a random walk on a uniformly elliptic time-dependent environment. The following hold Q_μ^c -a.s.*

- (i) *Let $C_3 > 0$. There exists a $t_0 > 0$ and constants C_1, C'_1 and C_2 such that for $\epsilon > 0$ small enough and every $t \geq t_0$, $y, z \in \mathbb{Z}^d$ such that $|y - z|_2 \leq \epsilon t + \frac{tC_3}{|\log \epsilon|}$ we have that*

$$C_1 e^{-C'_1 t \frac{1}{|\log \epsilon|^{1/2}}} p(\epsilon t, z, y) \leq e(t(1 - \epsilon), t, z, y) \leq C_2 e^{C_2 t \frac{1}{|\log \epsilon|^{1/2}}} p(\epsilon t, z, y).$$

- (ii) *Let $r > 0$. There exists a $t_0 > 0$ and a constant $C > 0$ such that for each $t \geq t_0$ and $x \in B_{tr}(0)$ one has that*

$$e(0, t, 0, x) \geq e^{-Ct} p(t, 0, x).$$

- (iii) *There is a function $\alpha : (0, \infty) \times [0, \infty) \rightarrow (0, \infty)$ such that for each $x, y \in \mathbb{Z}^d$ and $t > s \geq 0$ one has that*

$$e(s, t, x, y) \geq \alpha(t - s, |x - y|_1) > 0. \quad (2.5)$$

Proof. *Part (i).* Note that

$$\begin{aligned} & e(t(1 - \epsilon), t, z, y) \\ &= E_{z, t(1 - \epsilon)} \left[e^{\int_{t(1 - \epsilon)}^t \log(2d\omega_s(Y_{s-}, Y_s - Y_{s-})) dN_s - \int_{t(1 - \epsilon)}^t (\omega_s(Y_s, G) - 1) ds} \mathbf{1}_{Y_t}(y) \right], \quad (2.6) \end{aligned}$$

where $E_{z,s}$ is the expectation with respect to the law of a continuous time simple symmetric random walk $\{Y_t : t \geq 0\}$ of jump rate 1 starting from z at time s , N_t is the number of jumps up to time t of the walk, while for each $x \in \mathbb{Z}^d$ and $s > 0$, $\omega_s(x, G) := \sum_e \omega_s(x, e)$ is the total jump rate at site x and time s (see for example Proposition 2.6 in Appendix 1 of Kipnis-Landim [8]). Using the fact that the jump rates are bounded from above and from below, it is clear that there is a constant $C > 0$ such that

$$e^{\int_{t(1-\epsilon)}^t \log(2d\omega_s(Y_{s-}, Y_s - Y_{s-})) dN_s - \int_{t(1-\epsilon)}^t (\omega_s(Y_s, G) - 1) ds} \leq e^{C(N_t - N_{t(1-\epsilon)}) + C\epsilon t}.$$

Substituting this bound in (2.6), we see that

$$e(t(1-\epsilon), t, z, y) \leq e^{C\epsilon t} E \left[e^{CN_{\epsilon t}} p_{N_{\epsilon t}}(z, y) \right], \quad (2.7)$$

where now E is the expectation with respect to a Poisson process $\{N_t : t \geq 0\}$ of rate 1 and p_n is the n -step transition probability of a discrete time simple symmetric random walk. Let now $R_\epsilon := \frac{1}{\epsilon |\log \epsilon|^{1/2}}$. Note that

$$\begin{aligned} E \left[e^{CN_{\epsilon t}} p_{N_{\epsilon t}}(z, y) \right] &\leq e^{CR_\epsilon t \epsilon} p(\epsilon t, z, y) + E[e^{N_{\epsilon t} C}, N_{\epsilon t} > R_\epsilon t \epsilon] \\ &\leq e^{CR_\epsilon t \epsilon} p(\epsilon t, z, y) + E[e^{2N_{\epsilon t} C}]^{1/2} P(N_{\epsilon t} > R_\epsilon t \epsilon)^{1/2}. \end{aligned}$$

Now, using the exponential Chebychev inequality with parameter $\log R_\epsilon$, we get

$$P(N_{\epsilon t} > R_\epsilon \epsilon t) \leq e^{-\epsilon t (R_\epsilon \log R_\epsilon - (R_\epsilon - 1))} \quad (2.8)$$

and we compute $E[e^{2N_{\epsilon t} C}] = e^{\epsilon t (e^{2C} - 1)}$. Hence,

$$E \left[e^{CN_{\epsilon t}} p_{N_{\epsilon t}}(z, y) \right] \leq e^{CR_\epsilon t \epsilon} p(\epsilon t, z, y) + e^{\frac{\epsilon}{2} (e^{2C} - 1)} e^{-\epsilon \frac{\epsilon}{2} (R_\epsilon \log R_\epsilon - (R_\epsilon - 1))}. \quad (2.9)$$

Now, by Lemma 2.1 we know that $j(y)$ is increasing with $|y|$, so that

$$\begin{aligned} \sup_{y, z: |y-z|_2 \leq \epsilon t + \frac{C_3 t}{|\log \epsilon|}} \epsilon t j \left(\frac{|z-y|}{\epsilon t} \right) &\leq \epsilon t j \left(\frac{C_3}{\epsilon |\log \epsilon|} + 1 \right) \\ &\leq t \left(\frac{C_3}{|\log \epsilon|} + \epsilon \right) \log \left(3 + \frac{2C_3}{\epsilon |\log \epsilon|} \right) \end{aligned}$$

for $t \geq 1$. Hence, again by Lemma 2.1 with $\kappa = 1$, we see that for any

constant $c > 0$ we can choose ϵ small enough such that

$$\lim_{t \rightarrow \infty} \frac{e^{\frac{t}{2}(e^{2C}-1)} e^{-ctc(R_\epsilon \log R_\epsilon - (R_\epsilon - 1))}}{\inf_{y,z} p(\epsilon t, z, y)} = 0, \quad (2.10)$$

where the infimum is taken over y, z as in the previous display. Applying (2.10) with $c = 1/2$, we see that the second term of the right-hand side of inequality (2.9), after taking the supremum over y, z such that $|y - z|_2 \leq \epsilon t + \frac{C_3 t}{|\log \epsilon|}$, is negligible with respect to the first one. Hence, for ϵ small enough, there is a constant C and a $t_0 > 0$ such that for y, z such that $|y - z|_2 \leq \epsilon t + \frac{C_3 t}{|\log \epsilon|}$ and $t \geq t_0$ one has

$$e(t(1 - \epsilon), t, z, y) \leq C e^{(R_\epsilon + 1)Ct\epsilon} p(\epsilon t, z, y).$$

Similarly, using the fact that the jump rates are bounded from above and from below it can be shown that for y, z such that $|y - z|_2 \leq \epsilon t + \frac{C_3 t}{|\log \epsilon|}$ and t large enough

$$\begin{aligned} e(t(1 - \epsilon), t, z, y) &\geq e^{-C' \epsilon t} E[e^{-C' N_{\epsilon t}} p_{N_{\epsilon t}}(z, y) 1_{N_{\epsilon t} \leq R_\epsilon \epsilon t}] \\ &\geq e^{-(R_\epsilon + 1)\epsilon t C'} E[p_{N_{\epsilon t}}(z, y) 1_{N_{\epsilon t} \leq R_\epsilon \epsilon t}] \\ &\geq e^{-(R_\epsilon + 1)\epsilon t C'} (p(\epsilon t, z, y) - P(N_{\epsilon t} > R_\epsilon \epsilon t)) \\ &\geq C'' e^{-(R_\epsilon + 1)\epsilon t C'} p(\epsilon t, z, y), \end{aligned}$$

where we have used (2.8) and (2.10) with $c = 1$.

Part (ii). The proof of part (ii) is analogous to the proof of the lower bound of part (i).

Part (iii). By the same argument as the last part of the proof of part (i), there is a constant $C' > 0$ such that

$$e(s, t, x, y) \geq e^{-C'(t-s)} E[e^{-C' N_{t-s}} p_{N_{t-s}}(x, y), N_{t-s} = |x - y|_1]$$

But $P(N_{t-s} = |x - y|_1) > 0$ (there is, with positive probability, a trajectory from 0 to x such that $N_{t-s} = |x - y|_1$). Thus,

$$\begin{aligned} e(s, t, x, y) &\geq e^{-C'(t-s) - C'|x-y|_1} p_{|x-y|_1}(x, y) P(N_{t-s} = |x - y|_1) \\ &\geq e^{-C'(t-s) - C'|x-y|_1} \frac{1}{(2d)^{|x-y|_1}} P(N_{t-s} = |x - y|_1) > 0. \quad \square \end{aligned}$$

We can now apply Kingman's sub-additive ergodic theorem (see for example Liggett [9]), to prove the following lemma.

Lemma 2.3. *There exists a deterministic function $I_c : \mathbb{Q}^d \rightarrow [0, \infty)$ such that for every $y \in \mathbb{Q}^d$, Q_μ^c -a.s. we have that*

$$\lim_{\substack{t \rightarrow \infty \\ ty \in \mathbb{Z}^d}} \frac{a_c(0, t, 0, ty)}{t} = I_c(y). \quad (2.11)$$

Proof. Assume first that $y \in \mathbb{Z}^d$. Let $q \in \mathbb{N}$. We will consider for $m > n \geq 1$ the random variables

$$X_{n,m}(y) := a_c(nq, mq, ny, my).$$

By (2.3), we have

$$X_{0,m}(y) \leq X_{0,n}(y) + X_{n,m}(y).$$

By part (iii) of Lemma 2.2, we see that the random variables $\{X_{n,m}(y)\}$ are integrable. Hence, by Kingman's sub-additive ergodic theorem (see Liggett [9]) we can then conclude that the limit

$$\hat{I}(q, y, \omega) := \lim_{m \rightarrow \infty} \frac{a_c(0, mq, 0, my)}{m} \quad (2.12)$$

exists for $y \in \mathbb{Z}^d$ and $q \in \mathbb{N}$. We have to show that it is deterministic. For this reason, let $r > 0$, $z \in \mathbb{Z}^d$ be arbitrary. It suffices to prove that

$$\hat{I}(q, y, \omega) \leq \hat{I}(q, y, T_{r,z}\omega) = \lim_{m \rightarrow \infty} \frac{a_c(r, mq + r, z, my + z)}{m}.$$

First, we have that

$$\frac{a_c(0, mq, 0, my)}{m} \leq \frac{a_c(0, r, 0, z)}{m} + \frac{a_c(r, mq, z, my)}{m}.$$

By part (iii) of Lemma 2.2, the first term of the right-hand side of the last equation tends to 0 as $m \rightarrow \infty$. Therefore,

$$\hat{I}(q, y, \omega) = \lim_{m \rightarrow \infty} \frac{a_c(0, mq, 0, my)}{m} \leq \liminf_{m \rightarrow \infty} \frac{a_c(r, mq, z, my)}{m}. \quad (2.13)$$

On the other hand, for $u \in \mathbb{N}$ such that $m > u > r$ we have that

$$\frac{a_c(r, mq, z, my)}{m} \leq \frac{a_c(r, (m-u)q + r, z, (m-u)y + z)}{m} + \frac{a_c((m-u)q + r, mq, (m-u)y + z, my)}{m}.$$

Again, by part (iii) of Lemma 2.2, the last term tends to 0 as $m \rightarrow \infty$. Therefore

$$\begin{aligned} \liminf_{m \rightarrow \infty} \frac{a_c(r, mq, z, my)}{m} &\leq \lim_{m \rightarrow \infty} \frac{a_c(r, (m-u)q + r, z, (m-u)y + z)}{m} \\ &= \hat{I}(q, y, T_{r,z}\omega). \end{aligned} \quad (2.14)$$

Hence $\hat{I}(q, y, \omega) \leq \hat{I}(q, y, T_{r,z}\omega)$. Since $r > 0$ and $z \in \mathbb{Z}^d$ are arbitrary, $\hat{I}(q, y)$ is shift-invariant under each transformation $T_{r,z}$. By assumption (EC), $\hat{I}(q, y)$ is Q_μ^c -a.s. equal to a constant for each y . Now, if $y \in \mathbb{Q}^d$, choose the smallest $q \in \mathbb{N}$ such that $qy \in \mathbb{Z}^d$. Then by (2.12), we conclude that

$$\lim_{m \rightarrow \infty} \frac{a_c(0, mq, 0, mqy)}{mq} = \frac{1}{q} \hat{I}(q, qy, \omega) =: I_c(y), \quad (2.15)$$

exists (and is well-defined) and is Q_μ^c -a.s. equal to a constant. \square

We now need to extend the definition of the function $I_c(x)$ for all $x \in \mathbb{R}^d$ and prove the uniform convergence in (2.1). To do this, we will prove that for each compact K there is a $t_0 > 0$ such that the family of functions $\{t^{-1}a_c(0, t, 0, ty) : t \geq t_0\}$ defined on K is equicontinuous. We can now proceed to the main step of the proof of Theorem 2.1.

Lemma 2.4. *Let K be any compact subset of \mathbb{R}^d . There exist deterministic $\phi_K : (0, \infty) \rightarrow (0, \infty)$ with $\lim_{r \downarrow 0} \phi_K(r) = 0$, and $t_0 > 0$ such that for any $\epsilon > 0$ and $t \geq t_0$, Q_μ^c -a.s., we have*

$$\sup_{\substack{x, y \in tK \cap \mathbb{Z}^d \\ \|x-y\|_2 \leq \epsilon t}} t^{-1} |a_c(0, t, 0, x) - a_c(0, t, 0, y)| \leq \phi_K(\epsilon). \quad (2.16)$$

Proof. Let us note that for every $\epsilon > 0$, t and $x \in \mathbb{Z}^d$ one has that

$$e(0, t, 0, x) = \sum_{z \in \mathbb{Z}^d} e(0, t(1-\epsilon), 0, z) e(t(1-\epsilon), t, z, x).$$

Let $R_K := \sup\{|x|_2 : x \in K\}$ be the maximal distance to 0 for any point in K and $r_K = \frac{C_K}{|\log \epsilon|}$, where C_K is a constant that will be chosen large enough. From part (i) of Lemma 2.2 and Lemma 2.1, note that for $t \geq t_0$ (where t_0 is given by part (i) of Lemma 2.2)

$$\begin{aligned} e(0, t, 0, x) &\leq \sum_{z \in B_{r_K t}(x)} e(0, t(1 - \epsilon), 0, z) e(t(1 - \epsilon), t, z, x) \\ &\quad + C e^{\frac{1}{|\log \epsilon|^{1/2}} t C - \epsilon t \frac{1}{d} j(d \frac{r_K}{\epsilon})}. \end{aligned} \quad (2.17)$$

On the other hand by part (ii) of Lemma 2.2 we have that for $t \geq t_0$

$$e(0, t, 0, x) \geq e^{-C' t - t J(\frac{x}{t})}.$$

Using the upper bound $J(\frac{x}{t}) \leq d R_K \log(1 + 2d R_K)$ we see that if

$$\epsilon \frac{1}{d} j\left(d \frac{r_K}{\epsilon}\right) > C + C' + d R_K \log(1 + 2d R_K), \quad (2.18)$$

the second term of (2.17) is negligible. But (2.18) is satisfied for $C_K > 2(C + C' + d R_K \log(1 + 2d R_K))$ and $\epsilon > 0$ small enough. Hence, it is enough to prove that, Q_μ^c -a.s. we have that

$$\sup_{\substack{x, y \in tK \cap \mathbb{Z}^d \\ |x - y|_2 \leq \epsilon t}} \sup_{z \in B_{r_K t}(x)} \frac{e(t(1 - \epsilon), t, z, x)}{e(t(1 - \epsilon), t, z, y)} \leq C e^{t \phi_K(\epsilon)}. \quad (2.19)$$

To this end, by Lemmas 2.1 and 2.2

$$\frac{e(t(1 - \epsilon), t, z, x)}{e(t(1 - \epsilon), t, z, y)} \leq C e^{2tC \frac{1}{|\log \epsilon|^{1/2}}} e^{-\epsilon t (J(\frac{x-z}{\epsilon t}) - J(\frac{y-z}{\epsilon t}))}. \quad (2.20)$$

But,

$$\begin{aligned} J\left(\frac{z-x}{t\epsilon}\right) - J\left(\frac{z-y}{t\epsilon}\right) &= \sum_{i=1}^d \frac{1}{d} \left[j\left(d \frac{z_i - x_i}{t\epsilon}\right) - j\left(d \frac{z_i - y_i}{t\epsilon}\right) \right] \\ &\leq \sum_{i=1}^d \left| \frac{1}{d} \int_{d \frac{z_i - x_i}{t\epsilon}}^{d \frac{z_i - y_i}{t\epsilon}} \log(1 + 2|u|) du \right| \leq d \log\left(1 + \frac{2d C_K}{\epsilon |\log \epsilon|}\right). \end{aligned}$$

Substituting this estimate back into (2.20) we obtain (2.19) with $\phi_K(\epsilon) = C \frac{1}{|\log \epsilon|^{1/2}}$. \square

Using this lemma, we can extend I_c to a continuous function on \mathbb{R}^d . It remains to show the convexity of I_c . For this purpose, let $\lambda \in (0, 1)$, $x, y \in \mathbb{R}^d$ and let $(\lambda_n) \subset (0, 1) \cap \mathbb{Q}$, $(x_n), (y_n) \subset \mathbb{Q}^d$ such that $\lambda_n \rightarrow \lambda$, $x_n \rightarrow x$, and $y_n \rightarrow y$. In addition let $r_n \in \mathbb{N}$ be such that $r_n(\lambda_n x_n + (1 - \lambda_n)y_n)$, $\lambda_n m r_n$, and $\lambda_n m r_n x_n$, are contained in \mathbb{Z}^d . Then for any $n \in \mathbb{N}$ one has

$$\begin{aligned} & I_c(\lambda_n x_n + (1 - \lambda_n)y_n) \\ &= \lim_{m \rightarrow \infty} \frac{a_c(0, m r_n, 0, m r_n(\lambda_n x_n + (1 - \lambda_n)y_n))}{m r_n} \\ &\leq \lim_{m \rightarrow \infty} \frac{a_c(0, \lambda_n m r_n, 0, \lambda_n m r_n x_n)}{m r_n} \\ &\quad + \lim_{m \rightarrow \infty} \frac{a_c(\lambda_n m r_n, m r_n, \lambda_n m r_n x_n, m r_n(\lambda_n x_n + (1 - \lambda_n)y_n))}{m r_n}. \end{aligned}$$

Now taking $n \rightarrow \infty$, the continuity of I_c yields that the left-hand side converges to $I_c(\lambda x + (1 - \lambda)y)$. Taking advantage of the continuity of I_c and (2.15), the first summand on the right-hand side converges to $\lambda I_c(x)$ a.s., while in combination with the fact that the transformations $T_{\lambda_n m r_n, \lambda_n m r_n x_n}$ are measure preserving, the second summand converges in probability to $(1 - \lambda)I_c(y)$; from the last fact we deduce a.s. convergence along an appropriate subsequence and hence the convexity of I_c .

3. Proof of Theorem 1.2 for the convex case

Here we consider the case in which the jump range R of the walk is convex, symmetric and a neighborhood of 0 is contained in the convex hull of R . Let us call $\pi_{n,m}(x, y)$, the probability that the discrete time random walk in time-space random environment jumps from time n to time m from site x to site y . Define

$$a_d(n, m, x, y) := -\log \pi_{n,m}(x, y).$$

As in the continuous time case, we have the following sub-additivity property for $n \leq p \leq m$ and $x, y, z \in \mathbb{Z}^d$,

$$a_d(n, m, x, y) \leq a_d(n, p, x, z) + a_d(p, m, z, y). \quad (3.1)$$

We first need to define some concepts that will be used throughout this section. An element (n, z) of the set $\mathbb{N} \times \mathbb{Z}^d$ will be called a *time-space point*. The time-space points of the form $(1, z)$, with $z \in R$, will be called *steps*. Furthermore, given two time-space points $(n_1, x^{(1)})$ and $(n_2, x^{(2)})$ a sequence of steps $(1, z^{(1)}), \dots, (1, z^{(k)})$, with $k = n_2 - n_1$ will be called an *admissible path from $(n_1, x^{(1)})$ to $(n_2, x^{(2)})$* , if $x^{(2)} = x^{(1)} + z^{(1)} + \dots + z^{(k)}$ and

$$\begin{aligned} & \pi_{n_1, n_1+1}(x^{(1)}, x^{(1)} + z^{(1)}) \pi_{n_1+1, n_1+2}(x^{(1)} + z^{(1)}, x^{(1)} + z^{(1)} + z^{(2)}) \times \dots \\ & \dots \times \pi_{n_2-1, n_2}(x^{(1)} + z^{(1)} + \dots + z^{(k-1)}, x^{(1)} + z^{(1)} + \dots + z^{(k)}) > 0. \end{aligned} \quad (3.2)$$

In other words, there is a positive probability for the time-space random walk (n, X_n) to jump through the sequence of time-space points $(n_1, x^{(1)}), (n_1 + 1, x^{(1)} + z^{(1)}), \dots, (n_2, x^{(2)}) = (n_2, x^{(1)} + z^{(1)} + \dots + z^{(k)})$. Note that the sequence of steps $(1, z^{(1)}), \dots, (1, z^{(k)})$, is an admissible path if and only if $z^{(j)} \in R$ for all $1 \leq j \leq k$. Let us note that by uniform ellipticity asking that the left-hand side of (3.2) be positive is equivalent to asking that it be larger than or equal to $\kappa^{n_2 - n_1}$. With a slight abuse of notation, we will adopt the convention that for $u \in \mathbb{R}$, $[u]$ is the integer closest to u that is between u and 0. Furthermore, we introduce for $x \in \mathbb{R}^d$, the notation $[x] := ([x_1], \dots, [x_d]) \in \mathbb{Z}^d$. Throughout, given $A \subset \mathbb{R}^d$ we will call A° its interior.

Lemma 3.1. *Consider a discrete time random walk in a uniformly elliptic time-dependent environment ω with finite, convex and symmetric jump range R such that a neighborhood of 0 belongs to its convex hull. Then, U equals the convex hull of R and for every $n \geq 1$ we have that*

$$R_n = (nU) \cap \mathbb{Z}^d. \quad (3.3)$$

Proof. It is straightforward to check that U equals the convex hull of R in \mathbb{R}^d . On the other hand, note that if $x \in R_n$, we have that for every $m \in \mathbb{N}$, $mx \in R_{nm}$, which implies that $\frac{x}{n} \in U_{nm}$. This proves that $R_n \subset (nU) \cap \mathbb{Z}^d$. Finally, using the fact that R is convex, we can prove that $(nU) \cap \mathbb{Z}^d \subset R_n$. \square

For each $x \in \mathbb{Z}^d$ define $s(x)$ as the minimum number n of steps such that there is an admissible path between $(0, 0)$ and (n, x) . Alternatively,

$$s(x) = \min\{n \geq 0 : x \in R_n\}.$$

Let us now define a norm in \mathbb{R}^d which will be a good approximation for the previous quantity. For each $y \in \partial U$ define $\|y\| = 1$. Then, for each $x \in \mathbb{R}^d$ which is of the form $x = ay$ for some real $a \geq 0$, we define $\|x\| = a$. Note that since U is convex, symmetric and there is a neighborhood of 0 which belongs to its interior, this defines a norm in \mathbb{R}^d (see for example Theorem 15.2 of Rockafellar [16]) and that $x \in U^\circ$ if and only if $\|x\| < 1$. Furthermore, note that for every $x \in \mathbb{R}^d$ we have that

$$\|x\| \leq s(x) \leq \|x\| + 1. \quad (3.4)$$

Lemma 3.2. *Let $z \in U$ and $x \in U^\circ$. Then, for each natural n there exists an n_2 such that*

$$n \leq n_2 \leq n + \frac{4d+1}{1-\|x\|} + n \frac{\|x-z\|}{1-\|x\|} + 1. \quad (3.5)$$

and there is an admissible path between (n, z) and (n_2, x) so that

$$a_d(0, n_2, 0, [n_2x]) \leq a_d(0, n, 0, [nz]) - \log \kappa^{n_2-n}. \quad (3.6)$$

Similarly, whenever $\|z-x\| < 1-\|x\|$, there exists $n_0(x, z) = n_0$ such that for each natural $n \geq n_0$ there exists an n_1 such that

$$n - \frac{4d+1}{1-\|x\|} - n \frac{\|x-z\|}{\|1-x\|} - 1 \leq n_1 \leq n \quad (3.7)$$

and there is an admissible path between (n_1, x) and (n, z) so that

$$a_d(0, n, 0, [nz]) \leq a_d(0, n_1, 0, [n_1x]) - \log \kappa^{n-n_1} \quad (3.8)$$

Proof. We will first prove that there exists n_2 satisfying (3.5) and such that

$$s([n_2x] - [nz]) \leq n_2 - n. \quad (3.9)$$

Now, by (3.4) and the fact that $\|x - [x]\| \leq d$ we have that

$$\begin{aligned} s([n_2x] - [nz]) &\leq \| [n_2x] - [nz] \| + 1 \leq \| [n_2x] - [nx] \| + \| [nx] - [nz] \| + 1 \\ &\leq \| (n_2 - n)x \| + \| n(x - z) \| + 4d + 1 = (n_2 - n)\|x\| + n\|x - z\| + 4d + 1. \end{aligned}$$

It follows that (3.9) is satisfied whenever

$$(n_2 - n)\|x\| + n\|x - z\| + 4d + 1 \leq n_2 - n, \quad (3.10)$$

which is equivalent to

$$n_2 \geq n + \frac{4d + 1}{1 - \|x\|} + n \frac{\|x - z\|}{1 - \|x\|}.$$

This proves the first statement of the lemma, we will now show that there exists n_1 satisfying (3.7) such that

$$s([nz] - [n_1x]) \leq n - n_1.$$

Now,

$$s([nz] - [n_1x]) \leq |[nz] - [n_1x]| + 1 \leq n\|z - x\| + (n - n_1)\|x\| + 4d + 1.$$

Hence, it is enough to show that

$$n\|z - x\| + (n - n_1)\|x\| + 4d + 1 \leq n - n_1,$$

which is equivalent to

$$n_1 \leq n - \frac{4d + 1}{1 - \|x\|} - n \frac{\|z - x\|}{1 - \|x\|}. \quad \square$$

We are now ready to prove the following proposition.

Proposition 3.1. *For each $x \in \mathbb{R}^d$ we have that Q_μ^d -a.s. the limit*

$$I(x) := - \lim_{n \rightarrow \infty} \frac{1}{n} \log \pi_{0,n}(0, [nx]),$$

exists, is convex and deterministic. Furthermore, $I(x) < \infty$ if and only if $x \in U$.

Proof. From Lemma 3.1, it follows that for $x \notin U$ it is true for $n \geq 1$, that $nx \notin nU$ and hence from Lemma 3.1 that $nx \notin R_n$ so that $\pi_n(0, [nx]) = 0$. Thus, $I(x) = \infty$. We divide the rest of the proof in four steps. In step 1 for each $x \in \mathbb{Q}^d \cap U^\circ$ we define a function $\tilde{I}(x)$. In step 2 we will show

that \tilde{I} is deterministic for $x \in \mathbb{Q}^d \cap U^o$. In step 3 we will show that $I(x)$ is well-defined for $x \in \mathbb{Q}^d \cap U^o$ and that $I(x) = \tilde{I}(x)$ and in step 4, we extend the definition of $I(x)$ to $x \in U$.

Step 1. Here we will define for each $x \in \mathbb{Q}^d \cap U^o$ a function $\tilde{I}(x)$. Given $x \in \mathbb{Q}^d \cap U^o$, there exist a $k \in \mathbb{N}$ and a $y \in \mathbb{Z}^d \cap kU^o$ such that $x = k^{-1}y$. Now, by display (3.3) of Lemma 3.1 we know that $y \in R_k$. Then, by the convexity of R and the sub-additive ergodic theorem and (3.1) we can define \mathbb{Q}_μ^d -a.s.

$$\tilde{I}(k^{-1}y) := - \lim_{m \rightarrow \infty} \frac{1}{mk} \log \pi_{0,mk}(0, my).$$

This definition is independent of the representation of x . Indeed, assume that $x = k^{-1}y_1 = l^{-1}y_2$ for some $k, l \in \mathbb{N}$, $y_1 \in \mathbb{Z}^d \cap kU^o$ and $y_2 \in \mathbb{Z}^d \cap lU^o$. Then, passing to subsequences,

$$\begin{aligned} \tilde{I}(k^{-1}y_1) &= - \lim_{n \rightarrow \infty} \frac{1}{nlk} \log \pi_{0,nlk}(0, nly_1) \\ &= - \lim_{n \rightarrow \infty} \frac{1}{nlk} \log \pi_{0,nlk}(0, nky_2) = \tilde{I}(l^{-1}y_2). \end{aligned}$$

Step 2. Here we will show that \tilde{I} is deterministic in $\mathbb{Q}^d \cap U^o$. Let $x \in \mathbb{Q}^d \cap U^o$. We know that there exists a $k \in \mathbb{N}$ and a $y \in \mathbb{Z}^d \cap kU^o$ such that $x = k^{-1}y$. Let us now fix $z \in R$. It suffices to prove that

$$\tilde{I}(x, \omega) \leq \tilde{I}(x, T_{1,z}\omega) = \lim_{m \rightarrow \infty} \frac{a_d(1, mk + 1, z, my + z)}{mk}.$$

First, for each $n \in \mathbb{N}$, we have that

$$\frac{a_d(0, mnk, 0, mny)}{mnk} \leq \frac{a_d(0, 1, 0, z)}{mnk} + \frac{a_d(1, mnk, z, mny)}{mnk}.$$

By uniform ellipticity, the first term of the right-hand side of the last inequality tends to 0 as $m \rightarrow \infty$. Therefore,

$$\tilde{I}(x, \omega) = \lim_{m \rightarrow \infty} \frac{a_d(0, mnk, 0, mny)}{mnk} \leq \lim_{m \rightarrow \infty} \inf \frac{a_d(1, mnk, z, mny)}{mnk}. \quad (3.11)$$

On the other hand,

$$\frac{a_d(1, mnk, z, mny)}{mnk} \leq \frac{a_d(1, (m-1)nk + 1, z, (m-1)ny + z)}{mnk}$$

$$+ \frac{a_d((m-1)nk + 1, mnk, (m-1)ny + z, mny)}{mnk}. \quad (3.12)$$

Let us now assume that there is an admissible path from $(0, z + (m-1)ny)$ to $(nk-1, mny)$. This is equivalent to asking that z satisfies the following condition:

$$\pi_{0, nk-1}(z + (m-1)ny, mny) > 0 \quad \text{for some } n \in \mathbb{N}. \quad (3.13)$$

Then, by uniform ellipticity, the last term of (3.12) tends to 0 as $m \rightarrow \infty$. Therefore, if $z \in R$ satisfies condition (3.13), by (3.11) and (3.12) we have that

$$\tilde{I}(x, \omega) \leq \tilde{I}(x, T_{1,z}\omega). \quad (3.14)$$

Hence, to finish the proof it is enough to show that every $z \in R$ satisfies (3.13). Now, z satisfies (3.13) if and only if there exists an $n \in \mathbb{N}$ such that

$$z - ny \in R_{nk-1}. \quad (3.15)$$

We will show by contradiction that every $z \in R$ satisfies (3.15). Indeed, assume that for each n it is true that

$$z - ny \notin R_{nk-1}.$$

Then,

$$\frac{z}{nk-1} - y \frac{n}{nk-1} \notin U_{nk-1}.$$

Therefore, taking the limit $n \rightarrow \infty$, we conclude that $\frac{y}{k} \notin U^o$, which is a contradiction. This proves that for every $z \in R$ condition (3.13) is satisfied and hence (3.14) is also valid. It follows now by the ergodicity assumption (ED), that for each $x \in \mathbb{Q}^d \cap U^o$, $\tilde{I}(x)$ is Q_μ^d -a.s equal to a constant. *Step 3.* Here we will show that I is well-defined in $\mathbb{Q}^d \cap U^o$ and hence equals \tilde{I} there. Let $x \in \mathbb{Q}^d \cap U^o$. Let k be such that $kx \in \mathbb{Z}^d$. Given n , choose m so that $mk \leq n < (m+1)k$. Note that there exists a sequence of increments $z^{(j)} \in R$, $1 \leq j \leq n - mk$, such that

$$[nx] = mkx + z^{(1)} + \dots + z^{(n-mk)}.$$

Hence, by sub-additivity and considering that by uniform ellipticity the path $(1, z^{(1)}), \dots, (1, z^{(n-mk)})$ from $[nx]$ to mkx is admissible, we conclude that

$$\frac{a_d(0, n, 0, [nx])}{n} \leq \frac{a_d(0, mk, 0, mkx)}{n} - \frac{\log \kappa^{n-mk}}{n}.$$

It follows that

$$\limsup_{n \rightarrow \infty} \frac{a_d(0, n, 0, [nx])}{n} \leq \tilde{I}(x).$$

For the upper bound, first note that similarly there exists an admissible path of $(m+1)k - n$ steps from $[nx]$ to $(m+1)kx$. Hence,

$$\frac{a_d(0, (m+1)k, 0, (m+1)kx)}{n} \leq \frac{a_d(0, n, 0, [nx])}{n} - \frac{\log \kappa^{(m+1)k-n}}{n}.$$

Taking the limit when $n \rightarrow \infty$ we obtain

$$\liminf_{n \rightarrow \infty} \frac{a_d(0, n, 0, [nx])}{n} \geq \tilde{I}(x).$$

Step 4. Here we will show that I is well-defined in the set $(\mathbb{R}^d \setminus \mathbb{Q}^d) \cap U^o$. Let $z \in (\mathbb{R}^d \setminus \mathbb{Q}^d) \cap U^o$. Pick a rational point x such that $\|z - x\| < 1 - \|x\|$ and

$$\frac{1}{1 - \|x\|} \leq 2 \frac{1}{1 - \|z\|}. \quad (3.16)$$

From Lemma 3.2, for each $n \geq n_0$, we can find n_1, n_2 such that $n_1 \leq n \leq n_2$,

$$\frac{n_2}{n} \cdot \frac{1}{n_2} a_d(0, n_2, 0, [n_2 x]) \leq \frac{1}{n} a_d(0, n, 0, [nx]) + b \left(\frac{n_2}{n} - 1 \right)$$

and

$$\frac{1}{n} a_d(0, n, 0, [nz]) \leq \frac{n_1}{n} \cdot \frac{1}{n_1} a_d(0, n_1, 0, [n_1 x]) + b \left(1 - \frac{n_1}{n} \right),$$

where $b = -\log \kappa \in (0, \infty)$. Take $n \rightarrow \infty$. From (3.5) and (3.7) and taking $C(z) = 2 \frac{1}{1 - \|z\|}$, the limit points of $\frac{n_2}{n} - 1$ and $1 - \frac{n_1}{n}$ lie in the interval $[0, C(z)\|x - z\|]$ because x satisfies (3.16). Consequently from the last two inequalities we see that

$$I(x) \leq \liminf_{n \rightarrow \infty} \frac{1}{n} a_d(0, n, 0, [nz]) + C(z)b\|x - z\| \quad (3.17)$$

and

$$\limsup_{n \rightarrow \infty} \frac{1}{n} a_d(0, n, 0, [nz]) \leq I(x) + C(z)b||x - z||. \quad (3.18)$$

Letting $x \rightarrow z$, we conclude that I is well-defined in the set $(\mathbb{R}^d \setminus \mathbb{Q}^d) \cap U^o$. \square

We are now in a position to introduce the rate function of Theorem 1.2. We define, for each $x \in U$,

$$I_d(x) := \begin{cases} I(x) & \text{for } x \in U^o \\ \liminf_{U^o \ni y \rightarrow x} I(y) & \text{for } x \in \partial U \\ \infty & \text{for } x \notin U. \end{cases} \quad (3.19)$$

We will now prove that I_d satisfies the requirements of Theorem 1.2. By uniform ellipticity, it is clear that $I(x) \leq |\log \kappa|$ when $x \in U$. From (3.17) and (3.18), we see that I is continuous in the interior of R (in fact, Lipschitz continuous in any compact contained in U^o). These observations imply that I_d defined in (3.19) is bounded by $|\log \kappa|$ in U , is continuous in U^o , and is lower semi-continuous in U . The convexity of I_d is derived in a manner similar to the continuous time case. We now prove parts (i) and (ii) of Theorem 1.2.

Part (i) of Theorem 1.2 follows immediately from the definition of I_d and the fact that for open sets G , $\inf_{x \in G} I(x) = \inf_{x \in G} I_d(x)$. To prove part (ii) we first consider a compact set C contained in U^o . In this case, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{n} \log P_{0, \omega}^d \left(\frac{X_n}{n} \in C \right) &\leq \limsup_{n \rightarrow \infty} \sup_{x \in C} \frac{1}{n} \log \pi_{0, n}(0, [nx]) \\ &= \inf_n \sup_{m \geq n} \sup_{x \in C} \frac{1}{m} \log \pi_{0, m}(0, [mx]) \\ &= \inf_n \sup_{x \in C} \sup_{m \geq n} \frac{1}{m} \log \pi_{0, m}(0, [mx]) \\ &= \inf_n \sup_{x \in C} a_n(x), \end{aligned}$$

where we have defined for $x \in U^o$,

$$a_n(x) := \sup_{m \geq n} \frac{1}{m} \log \pi_{0, m}(0, [mx]).$$

Hence, the upper bound follows if we can show that, for any given $\epsilon > 0$,

$$\sup_{x \in C} a_n(x) \leq -\inf_{x \in C} I(x) + \epsilon$$

for large enough n . If we assume the opposite, we can find points $z_m \in C$ which have a subsequence converging to $z \in C$ and such that along this subsequence one also has that

$$\frac{1}{m} \log \pi_{0,m}(0, [mz_m]) > -I(z) + \epsilon.$$

Applying the first part of Lemma 3.2 gives an index $m_2 > m$ such that

$$\frac{1}{m_2} \log \pi_{0,m_2}(0, [m_2z]) \geq \frac{m}{m_2}(-I(z) + \epsilon) - b \left(1 - \frac{m}{m_2}\right).$$

Now, since $\lim_{m \rightarrow \infty} \frac{m}{m_2} = 1$ and since by Proposition 3.1 $\lim_{m_2 \rightarrow \infty} \frac{1}{m_2} \log \pi_{0,m_2}(0, [m_2z]) = -I(z)$, we obtain that $-I(z) \geq -I(z) + \epsilon$, which is a contradiction.

In the general case, let $C \subset U$ be a compact set. Fix $\delta > 0$ and let $C_1 = \frac{1}{1+\delta}C$. Now C_1 is a compact set contained in U° . Pick $\epsilon > 0$ small enough so that the closed ϵ -fattening $C_2 = \overline{C_1^{(\epsilon)}}$ is still a compact set contained in U° . Let $n_2 = \lfloor (1+\delta)n \rfloor$. Then for large enough n , $\frac{x}{n} \in C$ implies $\frac{x}{n_2} \in C_2$. By uniform ellipticity, we have that

$$\begin{aligned} P_{0,\omega}^d \left(\frac{X_n}{n} \in C \right) \kappa^{n_2-n} &= \sum_{x \in nC \cap \mathbb{Z}^d} P_{0,\omega}^d(X_n = x) \kappa^{n_2-n} \\ &\leq \sum_{x \in nC \cap \mathbb{Z}^d} P_{0,\omega}^d(X_n = x) \pi_{n,n_2}(x, x) \\ &= \sum_{x \in nC \cap \mathbb{Z}^d} P_{0,\omega}^d(X_n = x, X_{n_2} = x) \\ &\leq \sum_{x \in nC \cap \mathbb{Z}^d} P_{0,\omega}^d(X_{n_2} = x) \\ &\leq P_{0,\omega}^d \left(\frac{X_{n_2}}{n_2} \in C_2 \right), \end{aligned}$$

where the last inequality is satisfied for n large enough. Then, from the first

step of the proof of part (ii) of Theorem 1.2

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log P_{0,\omega}^d \left(\frac{X_n}{n} \in C \right) \leq - \inf_{x \in C_2} I(x) + \delta b.$$

By taking $\epsilon \searrow 0$ and using compactness and the continuity of I

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log P_{0,\omega}^d \left(\frac{X_n}{n} \in C \right) \leq - \inf_{x \in C_1} I(x) + \delta b.$$

Take $\delta \searrow 0$ along a subsequence δ_j . This takes C_1 to C . For each δ_j , let $z_j \in C_1 = C_1(\delta_j)$ satisfy $I(z_j) = \inf_{C_1(\delta_j)} I$. Pass to a further subsequence such that $\lim_{j \rightarrow \infty} z_j = z \in C$. Then regardless of whether z lies in the interior of U or not, by (3.19) $\liminf_{j \rightarrow \infty} I(z_j) \geq I_d(z) \geq \inf_C I_d$, and we get the final upper bound

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log P_{0,\omega}^d \left(\frac{X_n}{n} \in C \right) \leq - \inf_{x \in C} I_d(x).$$

4. Proof of Theorem 1.2 for the nearest neighbor case

Here we consider the case in which the jump range R of the random walk $\{X_n : n \geq 0\}$ is nearest neighbor. Define the *even lattice* as $\mathbb{Z}_{\text{even}}^d := \{x \in \mathbb{Z}^d : x_1 + \dots + x_d \text{ is even}\}$. Note that $\mathbb{Z}_{\text{even}}^d$ is a free Abelian group which is isomorphic to \mathbb{Z}^d . It therefore has a basis $f_1, \dots, f_d \in \mathbb{Z}_{\text{even}}^d$ and there is an isomorphism $h : \mathbb{Z}_{\text{even}}^d \rightarrow \mathbb{Z}^d$ such that $h(f_i) = e_i$ for $1 \leq i \leq d$. It is obvious that h can be extended as an automorphism defined in \mathbb{R}^d . Now, note that the random walk $\{Y_n : n \geq 0\}$ defined as

$$Y_n := h(X_{2n}),$$

is a random walk in \mathbb{Z}^d with finite, convex and symmetric jump range $Q = h(R)$ and such that a neighborhood of the origin is contained in its convex hull. From Theorem 1.2 for this class of random walks proved in Section 3, it follows that $\{Y_n : n \geq 0\}$ satisfies a large deviation principle with a rate function I . From this and the linearity of h we conclude that the limit

$$I_{\text{even}}(x) := I(h(x)) = - \lim_{n \rightarrow \infty} \frac{1}{2n} \log \pi_{0,2n}(0, h^{-1}([2nh(x)])), \quad (4.1)$$

exists Q_μ^d -a.s, where $\pi_{n,m}(x, y)$ is the probability that the random walk $\{X_n : n \geq 0\}$ jumps from time n to time m from site x to site y . Furthermore, if $U := \{x \in \mathbb{R}^d : |x| \leq 1\}$, as in (3.19), one can define

$$I_{d,even}(x) := \begin{cases} I_{even}(x) & \text{for } x \in U^\circ \\ \liminf_{U^\circ \ni y \rightarrow x} I_{even}(y) & \text{for } x \in \partial U \\ \infty & \text{for } x \notin U, \end{cases} \quad (4.2)$$

and $\{X_{2n} : n \geq 0\}$ satisfies a large deviation principle with rate function I_{even} .

At this point, we need to extend the above large deviation principle for the walk at even times, to all times taking into account the odd number of steps of the random walk. The next lemma will be very useful for this objective. To do this, we first prove that for each $x \in \mathbb{R}^d$ and each $g \in \mathcal{H} := \left\{ \sum_{i=1}^d c_i x : c_i \in \{-1, 0, 1\}, x \in R \right\}$ we have that,

$$I_{even}(x) := - \lim_{n \rightarrow \infty} \frac{1}{2n} \log \pi_{0,2n}(0, h^{-1}([2nh(x)]) + g) \quad Q_\mu^d\text{-a.s.} \quad (4.3)$$

Note that to prove (4.3), it is enough to show that for every $g \in \mathcal{H}$ we have that,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \tilde{\pi}_{0,n}(0, [nh(x)] + h(g)) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \tilde{\pi}_{0,n}(0, [nh(x)]), \quad (4.4)$$

where $\tilde{\pi}_{n,m}(x, y)$ is the probability that the random walk $\{Y_n : n \geq 1\}$ jumps from time n to time m from site x to site y . The proof that the limit in the right-hand side of (4.4) exists, is a repetition of the proofs of Lemma 3.2 and Proposition 3.1, so we omit it. We just point out here that in the proof of Lemma 3.2 we need to replace the points $[nz]$, $[n_1x]$ and $[n_2x]$ by $[nz] + h$, $[n_1x] + h$ and $[n_2x] + h$ respectively. On the other hand, the equality in (4.4) is established using the uniform ellipticity of the walk and the Markov property.

Let us now see how to derive from (4.3) the large deviation principle for a random walk with a nearest neighbor jump range R . Note that for any

subset $A \subseteq \mathbb{R}^d$ one has that

$$\begin{aligned} P_{0,\omega} \left(\frac{X_{2n+1}}{2n+1} \in A \right) &= \sum_{i=1}^{2d} \pi_{0,1}(0, e_i) P_{e_i, \omega} \left(\frac{X_{2n}}{2n} \in A \right) \\ &= \sum_{i=1}^{2d} \pi_{0,1}(0, e_i) P_{0, \bar{\omega}} \left(\frac{X_{2n}}{2n} \in A - \frac{e_i}{2n} \right) \end{aligned}$$

where $\bar{\omega} = \{\omega_n : n \geq 1\}$ and $e_{i+d} = -e_i$ for $i = 1, \dots, d$. We will show that $P_{e_i, \omega} \left(\frac{X_{2n}}{2n} \in A \right)$ does not depend on e_i , regardless of whether A is an open subset or a closed subset of \mathbb{R}^d and we will use the result obtained in the even case. It is important to note that this argument can be used, even with $\bar{\omega}$, because the limit depends only on the distribution of ω .

Now, when $A = G$, where G is an open subset of \mathbb{R}^d , we can follow the arguments used in the convex case, observing that for any $x \in G$ and any $i \in \{1, \dots, d\}$, $[nx] + e_i \in nG$, for n large enough. On the other hand, if $A = C$, where C is a compact subset of U_2° , note that

$$\begin{aligned} &\limsup_{n \rightarrow \infty} \frac{1}{2n} \log P_{0, \bar{\omega}} \left(\frac{X_{2n}}{2n} \in C - \frac{e_i}{2n} \right) \\ &\leq \limsup_{n \rightarrow \infty} \sup_{x \in C - \frac{e_i}{2n}} \frac{1}{2n} \log \pi_{0, 2n}(0, h^{-1}([2nh(x)])) \\ &= \limsup_{n \rightarrow \infty} \sup_{x \in C} \frac{1}{2n} \log \pi_{0, 2n}(0, h^{-1}([2nh(x) - h(e_i)])) \\ &\leq \limsup_{n \rightarrow \infty} \sup_{x \in C} \max_{g \in \mathcal{H}} \frac{1}{2n} \log \pi_{0, 2n}(0, h^{-1}([2nh(x)])) + g \end{aligned}$$

However, by (4.3) the last expression is independent of g .

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