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EXACT DIFFERENCE TRIANGLES

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Abstract. The non-existence of exact difference triangles of order greater than five is demonstrated.

1. **Introduction.** For any positive integer $n \geq 2$, let

$$\nabla_n = \{(i, j) \mid 1 \leq j \leq i \leq n\}.$$

An integral-valued function X on ∇_n is called a *difference triangle* of order n if the following conditions are satisfied.

(i) X is injective,
(ii) $|X((i+1, j)) - X((i+1, j+1))| = X((i, j))$ for $1 \leq j \leq i \leq n-1$.

X is said to be *exact* if $1 \leq X((i, j)) \leq \frac{1}{2}n(n+1)$ for all $(i, j) \in \nabla_n$.

We write $x_{i,j}$ for $X((i, j))$. The diagram below is called the *graph* of X.

$$\begin{array}{c} x_{n,1} \quad x_{n,2} \quad \cdots\cdots\cdots\cdots \quad x_{n,n-1} \quad x_{n,n} \\ x_{n-1,1} \quad x_{n-1,2} \quad \cdots \quad x_{n-1,n-2} \quad x_{n-1,n-1} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ x_{3,1} \quad x_{3,2} \quad x_{3,3} \\ x_{2,1} \quad x_{2,2} \\ x_{1,1} \end{array}$$

Although difference triangles are abundant, only a handful of them are exact. Examples.

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 3  2        3  1
   1           2

 6  2  5     2  6  5     6  1  4     1  6  4
   4  3        4  1        5  3        5  2
     1           3           2           3

 6  1 10  8     6 10  1  8     8  3 10  9     8 10  3  9
   5  9  2        4  9  7        5  7  1        2  7  6
     4  7           5  2           2  6           5  1
       3              3              4              4

 13  3 15 14  6
   10 12  1  8
      2 11  7
        9  4
          5
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Up to reflection, the graphs exhibited above exhaust all exact difference triangles of order not greater than five. We will show that there exist no exact difference triangles of order greater than five. This solves the general pool-ball problem posed by M. Gardner [1].

2. b_j and s_j. From now on let X be an exact difference triangle of order $n \geq 4$. Let

$$S_n = \{1, 2, \cdots, n\},$$
$$B_n = \{\tfrac{1}{2}n(n+1) - j \mid j = 0, 1, \cdots, n\},$$
$$R_k = \{x_{k,1}, x_{k,2}, \cdots, x_{k,k}\}, \ k = 1, 2, \cdots, n.$$

Note that $S_n \cap B_n = \emptyset$ and $|b - b'| \in S_n$ if b and b' are distinct elements of B_n.

For each k, $1 \leq k \leq n$, we now choose a pair of numbers b_k and s_k from R_k by the following inductive method.

(i) $b_1 = s_1 = x_{1,1}$;

(ii) if $b_j, s_j \in R_j$, $1 \le j \le n-1$, have already been chosen, then $b_j = x_{j,h}$ for some h, $1 \le h \le j$. Let

$$b_{j+1} = \max\{x_{j+1,h},\ x_{j+1,h+1}\},$$
$$s_{j+1} = \min\{x_{j+1,h},\ x_{j+1,h+1}\}.$$

LEMMA 1. (i) *$b_i < b_j$ if $1 \le i < j \le n$.*

(ii) $S_n = \{s_1, s_2, \cdots, s_n\}$.

(iii) $\#(R_j \cap S_n) = 1$ *for* $j = 1, 2, \cdots, n$.

Proof. By way of defining b_k and s_k, we have $b_{k+1} - s_{k+1} = b_k$ for $1 \le k \le n-1$. Suppose $i < j$. Then

$$\sum_{k=i}^{j-1} b_{k+1} - \sum_{k=i}^{j-1} s_{k+1} = \sum_{k=i}^{j-1} b_k.$$

So

$$b_j = b_i + \sum_{k=i+1}^{j} s_k > b_i.$$

In particular, we have

$$b_n = b_1 + \sum_{k=2}^{n} s_k = \sum_{k=1}^{n} s_k.$$

The s_k's are pairwise distinct positive integers. Thus

$$\frac{1}{2} n(n+1) \ge b_n = \sum_{k=1}^{n} s_k \ge 1 + 2 + \cdots + n = \frac{1}{2} n(n+1).$$

Consequently, $b_n = \frac{1}{2} n(n+1)$, $S_n = \{s_1, s_2, \cdots, s_n\}$, and $\#(R_j \cap S_n) = \#\{s_j\} = 1$.

LEMMA 2. *For all j, $s_j = \min R_j$ and $b_j = \max R_j$.*

Proof. Since $R_j \cap S_n = \{s_j\}$, all elements of R_j except s_j belong to $S_{n(n+1)/2} - S_n$ and are $> n \ge s_j$. Therefore $s_j = \min R_j$.

We have already seen

$$b_n = \frac{1}{2} n(n+1) = \max R_n.$$

Now suppose $j < n$. By induction we may assume $y \leq b_{j+1}$ for all $y \in R_{j+1}$. Let x be an arbitrary element in R_j. There are $u_{j+1}, v_{j+1} \in R_{j+1}$ such that

$$x = u_{j+1} - v_{j+1} \leq b_{j+1} - s_{j+1} = b_j.$$

Hence $b_j = \max R_j$.

LEMMA 3. *For all* j, $b_j \leq \frac{1}{2} j(2n - j + 1)$.

Proof. $b_j = b_n - \sum_{k=j+1}^{n} s_k$

$$\leq \frac{1}{2} n(n+1) - (1 + 2 + \cdots (n-j))$$

$$= \frac{1}{2} j(2n - j + 1).$$

3. **Distribution of B_n.** Assume that, for some $k \leq n-1$, there exists $\{x_{k,i}, x_{k,j}\} \subset B_n$, where $i < j$. Then

$$\min\{x_{k+1,i},\ x_{k+1,i+1}\} \in S_n,$$
$$\min\{x_{k+1,j},\ x_{k+1,j+1}\} \in S_n,$$
$$\max\{x_{k+1,i},\ x_{k+1,i+1}\} \in B_n,$$
$$\max\{x_{k+1,j},\ x_{k+1,j+1}\} \in B_n.$$

Since $\#(R_{k+1} \cap S_n) = 1$, we must have

$$\min\{x_{k+1,i},\ x_{k+1,i+1}\} = \min\{x_{k+1,j},\ x_{k+1,j+1}\}.$$

This can happen only when $j = i+1$. Thus $x_{k+1,i+1} = x_{k+1,j} \in S_n$. We also have $x_{k+1,i} \in B_n$ and $x_{k+1,i+2} \in B_n$. This latter fact would contradict the result just obtained unless $k+1 \nleq n-1$, i.e. $k = n-1$.

The above discussion gives us the following

LEMMA 4. (i) $\#(R_k \cap B_n) \leq 1$ *if* $k \leq n-2$;

(ii) $\#(R_{n-1} \cap B_n) \leq 2$;

(iii) *if* $\#(R_{n-1} \cap B_n) = 2$, *then there is* $i < n-1$ *such that* $x_{n,i+1} \in S_n$ *and*

$$\{x_{n-1,i},\ x_{n-1,i+1},\ x_{n,i},\ x_{n,i+2}\} \subset B_n.$$

Two observations about the top row R_n:

(1) If $\{x_{n,j},\ x_{n,j+1}\}\subset B_n$, then $x_{n-1,j}=|x_{n,j}-x_{n,j+1}|\in S_n$. Thus there is no $\{x_{n,i},\ x_{n,i+1},\ x_{n,j},\ x_{n,j+1}\}\subset B_n$ with $1\leq i\neq j\leq n-1$.

(2) If $\{x_{n,j},\ x_{n,j+2}\}\subset B_n$ and $x_{n,j+1}\notin B_n$, then $x_{n-2,j}=|x_{n,j}-x_{n,j+2}|\in S_n$. Thus there is no $\{x_{n,i}, x_{n,i+2}, x_{n,j}, x_{n,j+2}\}\subset B_n$ with $1\leq i\neq j\leq n-2$.

From (1) and (2), it follows easily that

LEMMA 5. $\#(R_n\cap B_n)\leq\frac{1}{3}(n+5)$.

Now we assume temporarily $\#(R_{n-1}\cap B_n)=2$. So the situation described in Lemma (iii) does happen. In the graph of X, b_j and s_j are adjacent when $j>1$. Therefore $b_n=x_{n,i}$ or $b_n=x_{n,i+2}$. Without loss of generality, we may assume $b_n=x_{n,i+2}$. Then $b_{n-1}=b_n-s_n=x_{n-1,i+1}$. Since s_{n-1} is adjacent to b_{n-1} and $x_{n-1,i}\notin S_n$, we have $s_{n-1}=x_{n-1,i+2}$ and then $x_{n,i+3}=b_n-s_{n-1}\in B_n$. If we let $*$ stand for a position occupied by an element of B_n, then a part of the graph of X appears as follows.

$$\begin{array}{ccccccc} * & & s_n=x_{n,i+1} & & b_n=x_{n,i+2} & & * \\ & * & & b_{n-1}=x_{n-1,i+1} & & s_{n-1} & \\ & & s_{n-2} & & & & \end{array}$$

Now let $u=\frac{1}{2}n(n-1)-1$. If $u\in R_k$ for some $k\leq n-2$, then, since $b_n\notin R_{k+1}$, u must lie right below a certain element of B_n and a certain element of S_n. The argument used in the proof of Lemma 4 can be adapted to show

LEMMA 6. $\#(R_k\cap(B_n\cup\{u\}))\leq 1$ *for* $k\leq n-2$.

If $u=x_{n-1,k}\in R_{n-1}$, then $\max\{x_{n,k},\ x_{n,k+1}\}\neq b_n$. The number $\min\{x_{n,k},\ x_{n,k+1}\}$, which is not s_n, must belong to S_n. This is absurd. So $u\notin R_{n-1}$. Next assume that $u=x_{n,k}\in R_n$. Among all elements of $R_n\cap B_n$, let $b=x_{n,j}$ be closest to u in the graph of X. Clearly $b\neq b_n$. If $|j-k|=1$, then $b-u\neq s_{n-1}$ lies in $R_{n-1}\cap S_n$. If $|j-k|=2$, then $b-u\neq s_{n-2}$ lies in $R_{n-2}\cap S_n$. Both are impossible. Summarizing these results, we obtain

LEMMA 7. *Let* $u = \frac{1}{2} n(n-1) - 1$ *and* $\# (R_{n-1} \cap B_n) = 2.$ *Then*

(i) $u \notin R_{n-1}$;

(ii) *if* $u \in R_n$, *then* u *must be separated from* $R_n \cap B_n$ *by at least two elements of* R_n.

4. **Main theorem.** Now we are ready to prove the main result.

THEOREM. *Let* X *be an exact difference triangle of order* n. *Then* $n \leq 5$.

Proof. Let p be the smallest positive integer such that $b_p \geq \frac{1}{2} n(n-1)$. By Lemma 3,

$$\frac{1}{2} p(2n - p + 1) \geq b_p \geq \frac{1}{2} n(n-1).$$

We have

$$2n \geq (n-p)^2 + (n-p). \tag{1}$$

On the other hand, we know

$$n + 1 = \# B_n = \sum_{k=1}^{n} \# (R_k \cap B_n)$$

$$= \sum_{k=p}^{n} \# (R_k \cap B_n)$$

$$\leq \frac{1}{3} (n+5) + 2 + (n-p-1).$$

Thus

$$3(n-p) + 5 \geq 2n. \tag{2}$$

(1) and (2) together yield

$$(n-p)^2 - 2(n-p) - 5 \leq 0.$$

If follows that $n - p \leq 3$ and $2n \leq 3(n-p) + 5 \leq 14$, i. e. $n \leq 7$.

For $n = 6$ and $n = 7$, we see that $\# (R_n \cap B_n) \leq$ the integral part of $\frac{1}{3} (n+5)$, i. e. $n - 3$. Now $2 \geq \# (R_{n-1} \cap B_n) \geq (n+1) - (n-3) - (n-p-1) = 5 - (n-p) \geq 2$. Therefore we must have $n - p = 3$ and

$$\# (R_n \cap B_n) = n - 3,$$
$$\# (R_{n-1} \cap B_n) = 2,$$
$$\# (R_{n-2} \cap B_n) = 1,$$
$$\# (R_{n-3} \cap B_n) = 1.$$

Note that, for $n = 6$ and 7, $b_{n-4} \leq \frac{1}{2}(n-4)(n+5) < u = \frac{1}{2}n(n-1) - 1$. By Lemmas 6 and 7, u must appear on the top row. Let us count the number of elements in R_n. Besides the $n-3$ elements of $R_n \cap B_n$, there are s_n, u, and at least two other elements to separate u from $R_n \cap B_n$. Thus

$$n = \# R_n \geq (n-3) + 2 + 2 = n + 1.$$

This is impossible. Hence $n \neq 6$ and $n \neq 7$. Our theorem is proved.

Reference

1. M. Gardner, *Mathematical games*, Scientific American **236** (1977), No. 4, 129–136.

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