ON EXISTENCE OF WILLIAMSON SYMMETRIC CIRCULANT MATRICES

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Abstract

In this paper we consider a particular type of partition of $\mathbb{Z}_n$, called $H$-partition and obtain a necessary and sufficient condition for existence of a set of four symmetric circulant matrices for a Hadamard matrix of order $4n$ in terms of such partitions when $n$ odd.

1. Introduction

A $(1, -1)$ matrix $H$ of order $n$ is called a Hadamard matrix if $HH' = nI$, where $H'$ is the transpose of $H$. If $H$ is a Hadamard matrix of order $n$ then $n = 2$ or $n \equiv 0 \pmod{4}$. The converse of this seems to be true and is known as Hadamard conjecture.

Many exciting results have stemmed from the following basic idea put forward by Williamson. Consider the array

$$H = \begin{pmatrix}
W & X & Y & Z \\
-X & W & -Z & Y \\
-Y & Z & W & -X \\
-Z & -Y & X & W
\end{pmatrix}$$

If $W, X, Y, \text{ and } Z$ are replaced by square matrices $A, B, C, \text{ and } D$ of order $n$, respectively, then $H$ becomes a square matrix of order $4n$. Williamson proved that a sufficient condition for $H$ to be a Hadamard matrix is that

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\(A, B, C, \text{ and } D\) are \((1, -1)\) matrices of order \(n\) with
\[
AA' + BB' + CC' + DD' = 4nI \tag{1}
\]
and for every pair \(X, Y\) of matrices chosen from \(A, B, C, D\)
\[
XY' = YX' \tag{2}
\]
If \(A, B, C, \text{ and } D\) are symmetric and circulant then condition \((2)\) is satisfied trivially and condition \((1)\) becomes
\[
A^2 + B^2 + C^2 + D^2 = 4nI \tag{3}
\]
The basic difficulty lies in finding the matrices \(A, B, C, \text{ and } D\) which satisfy the condition \((3)\). In this article we give a necessary and sufficient condition for the existence of such symmetric circulant matrices \(A, B, C, \text{ and } D\). Our result also gives a method for finding a set of such matrices.

2. Definitions

**Definition 2.1.** For any odd integer \(n\), let \(\mathbb{Z}_n\) be the cyclic group of integers modulo \(n\) under addition. Let \(A\) be a proper subset of \(\mathbb{Z}_n\) such that \(0 \in A\) and \(A = -A\). Then \(A, B = \mathbb{Z}_n - A\) is clearly a partition of \(\mathbb{Z}_n\) such that \(B = -B\). We call such a partition of \(\mathbb{Z}_n\) to be an \(H\)-partition of \(\mathbb{Z}_n\).

For an \(H\)-partition \((A, B)\) of \(\mathbb{Z}_n\), let \(A + B = \{a + b (\text{mod } n) \mid a \in A, b \in B\}\). Let \(C\) denote the set of distinct elements of \(A + B\). For any \(c \in C\) we denote \(n_c\) the frequency of occurrence of \(c\) in \(A + B\). Clearly \(0 \notin C\) for any \(H\)-partition \((A, B)\) of \(\mathbb{Z}_n\).

**Definition 2.2.** A set of 4 symmetric circulant matrices \(A, B, C, \text{ and } D\) satisfying the condition \(A^2 + B^2 + C^2 + D^2 = 4nI\) is called a set of Williamson circulant matrices.

**Definition 2.3.** The *shift matrix* \(T\) of order \(n\) is a \((0,1)\)-square matrix defined as \(T = [u_{ij}]\), where
\[
u_{ij} = \begin{cases} 
 1, & \text{if } j - i \equiv 1 \text{ mod } n; \\
 0, & \text{otherwise}.
\end{cases}
\]
Definition 2.4. For any matrix $A$, the Match matrix $A^{(m)}$ of $A$ is defined as $A^{(m)} = [n_{ij}]$, where $n_{ij}$ = number of places in which the $i^{th}$ row and $j^{th}$ row of $A$ have same non-zero entry at corresponding places.

Definition 2.5. For any matrix $A$ with nonzero entries, the Mis-match matrix $A^{(mm)}$ of $A$ is defined to be $A^{(mm)} = [n_{ij}]$, where $n_{ij}$ = number of places in which the $i^{th}$ row and $j^{th}$ row of $A$ have different entries at corresponding places.

Definition 2.6. Let $a_0, a_1, \ldots, a_{n-1}$ be a sequence of $n$ elements then a matrix $C = [c_{ij}]$ is called a Circulant matrix with entries $a_0, a_1, \ldots, a_{n-1}$ if $c_{ij} = a_{(j-i) \mod n}$; for $1 \leq i, j \leq n$.

Clearly $C$ is a circulant matrix if and only if $C = \sum_{i=0}^{n-1} a_i T^i$.

We now have the following result.

3. Result

Theorem 3.1. There exists a set of four Williamson symmetric circulant matrices of order $n$ if and only if there exists four $H$-partitions $(A_i, B_i)$, $i = 1, 2, 3, 4$, of $\mathbb{Z}_n$, not necessarily distinct, such that $\bigcup_{i=1}^{4} C_i = \mathbb{Z}_n - \{0\}$ and $\sum_{i=1}^{4} n_i^c = n$ for each $c \in \mathbb{Z}_n - \{0\}$ where $n_i^c$ denotes the occurrence number of $c$ in $A_i + B_i$.

Proof. Let $T$ be the shift matrix of order $n$. For any set of four $H$-partitions $(A_i, B_i)$, $i = 1, 2, 3, 4$ of $\mathbb{Z}_n$ of the stated type, let $P_i = \sum_{a_i \in A_i} T^{a_i}$ and $N_i = \sum_{b_i \in B_i} T^{b_i}$. Then $P_i$ and $N_i$ are symmetric circulant $(0, 1)$ matrices and $P_i N_i = \sum_{c \in C_i} n_i^c T^c$

$$\Rightarrow \sum_{i=1}^{4} P_i N_i = n \sum_{c \in \mathbb{Z}_n - \{0\}} T^c = n(J - I) \quad (1)$$
Now let $X_i = P_i - N_i$ for $i = 1, 2, 3, 4$; then $X_i's$ are symmetric circulant matrices with entries 1 and $-1$ and hence $X_i$, $X_j$ commutes for $i, j \in \{1, 2, 3, 4\}$. From Definition 2.4 it is clear that for a symmetric $(0, 1)$-matrix $A$ the match matrix $A^{(m)} = A^2$. Since $P_i$’s and $N_i$’s are symmetric $(0, 1)$-matrices, $P_i^{(m)} = P_i^2$ and $N_i^{(m)} = N_i^2$; and $X_i^{(m)} = P_i^{(m)} + N_i^{(m)}$ for $i = 1, 2, 3, 4$; since $(A_i, B_i)$ is a partition of $\mathbb{Z}_n$. So

$$X_i^{(m)} = P_i^2 + N_i^2; i = 1, 2, 3, 4; \quad (2)$$

From Definition 2.5 it is clear that, for a $(1, -1)$-matrix $A$ of order $n$, the mis-match matrix $A^{(mm)} = [\hat{n}_{ij}] = [n - n_{ij}]$, where $n_{ij}$ is the $(i, j)^{th}$ entry of $A^{(m)}$.

Therefore $A^{(mm)} = nJ - A^{(m)}$, where $J$ is the square matrix with entry 1. Since $X_i$ is a $(1, -1)$-matrix,

$$X_i^{(mm)} = nJ - X_i^{(m)}$$

$$\Rightarrow X_i^{(mm)} = nJ - (P_i^2 + N_i^2); i = 1, 2, 3, 4 \quad (3)$$

Also, since $X_i$ is a symmetric $(1, -1)$-matrix $X_i^2 = [x_{kl}]$, where $x_{kl} =$ inner product of the $k^{th}$ row and $l^{th}$ row of $X_i =$ (number of places in which the $k^{th}$ row and $l^{th}$ row of $X_i$ have the same entries) - (number of places in which the $k^{th}$ row and $l^{th}$ row of $X_i$ have different entries).

Thus

$$X_i^2 = X_i^{(m)} - X_i^{(mm)}$$

$$\Rightarrow X_i^2 = 2(P_i^2 + N_i^2) - nJ; i = 1, 2, 3, 4$$

$$\Rightarrow \sum_{i=1}^{4} X_i^2 = 2(\sum_{i=1}^{4} P_i^2 + \sum_{i=1}^{4} N_i^2) - 4nJ \quad (4)$$

Again

$$\sum_{i=1}^{4} X_i^2 = \sum_{i=1}^{4} (P_i - N_i)^2$$

$$= \sum_{i=1}^{4} P_i^2 + \sum_{i=1}^{4} N_i^2 - 2 \sum_{i=1}^{4} P_i N_i$$
\[ \Rightarrow \sum_{i=1}^{4} P_i^2 + \sum_{i=1}^{4} N_i^2 = \sum_{i=1}^{4} X_i^2 + 2 \sum_{i=1}^{4} P_i N_i \] (5)

From equations (4) and (5)
\[ \sum_{i=1}^{4} X_i^2 = 2 \left( \sum_{i=1}^{4} X_i^2 + 2 \sum_{i=1}^{4} P_i N_i \right) - 4nJ \]
\[ \Rightarrow \sum_{i=1}^{4} X_i^2 = 4nJ - 4 \sum_{i=1}^{4} P_i N_i \] (6)

So equations (4) and (6) imply
\[ \sum_{i=1}^{4} X_i^2 = 4nJ - 4n(J - I) \]
\[ = 4nI \]

Thus $X_i$, $i = 1, 2, 3, 4$ form a set of four Williamson circulant matrices for a Hadamard matrix of order $4n$.

Conversely, let $X_i$, $i = 1, 2, 3, 4$ be a set of four Williamson symmetric circulant matrices of order $n$. Then
\[ \sum_{i=1}^{4} X_i^2 = 4nI \] (7)
and
\[ X_i X_j = X_j X_i, \] (8)
for $i, j = \{1, 2, 3, 4\}$.

Since $X_i$ is a $(1, -1)$ circulant matrix, it can be written as
\[ X_i = \sum_{k=0}^{n-1} a_k T^k; \ a_i = \pm 1 ; i = 1, 2, 3, 4 \] (9)

Let $A_i = \{k, k \in \mathbb{Z}_n \mid a_k = +1\}$ and $B_i = \{k, k \in \mathbb{Z}_n \mid a_k = -1\}$, then clearly $(A_i, B_i)$, $i = 1, 2, 3, 4$ are four partitions of $\mathbb{Z}_n$ and exactly one of $A_i$ and $B_i$ contains 0. Since equation (7) remains valid if $X_i$ is replaced by $-X_i$, replacing $X_i$ by $-X_i$, if necessary, we can assume that $A_i$ contains 0, for $i = 1, 2, 3, 4$. As $\pm X_i$ is a symmetric circulant matrix $k \in A_i \Rightarrow n-k \in A_i$
and so \((A_i, B_i), i = 1, 2, 3, 4\) are four \(H\)-partitions of \(\mathbb{Z}_n\). Let \(P_i = \sum_{k \in A_i} T^k\) and \(N_i = \sum_{k \in B_i} T^k\). Then \(X_i = P_i - N_i; i = 1, 2, 3, 4\) and \(P_i\) and \(N_i\) are symmetric matrices with entries \((0, 1)\). Thus \(P_i^{(m)} = P_i^2\) and \(N_i^{(m)} = N_i^2\), and \(X_i^{(m)} = P_i^{(m)} + N_i^{(m)}\) for \(i = 1, 2, 3, 4\). So

\[
X_i^{(m)} = P_i^2 + N_i^2; \quad i = 1, 2, 3, 4. \tag{10}
\]

Since \(X_i\) is a \((1, -1)\)-matrix from Definition 2.5

\[
X_i^{(mm)} = nJ - X_i^{(m)}. \tag{11}
\]

Using equations (7), (10) and (11) we get

\[
\sum_{i=1}^{4} P_i N_i = n(J - I) \tag{12}
\]

Now, if possible, let us assume that for some element \(k \in \mathbb{Z}_n - \{0\}, \sum_{i=1}^{4} n_i^k = n_k \neq n\). As \(P_i N_i = \sum_{c \in C_i} n_c^i T^c; i = 1, 2, 3\) and 4, where \(C_i\) is the set determined by \(A_i + B_i\).

\[
\sum_{i=1}^{4} P_i N_i = \sum_{i=1}^{4} (\sum_{c \in C_i} n_c^i T^c) = \sum_{c \in C} (\sum_{i=1}^{4} n_c^i) T^c, \quad \text{where} \quad C = \bigcup_{i=1}^{4} C_i
\]

\[
= \sum_{c \in C - \{k\}} (\sum_{i=1}^{4} n_c^i) T^c + \sum_{i=1}^{4} n_k^i T^k = \sum_{c \in C - \{k\}} (\sum_{i=1}^{4} n_c^i) T^c + n_k T^k
\]

But this contradicts

\[
\sum_{i=1}^{4} P_i N_i = n(J - I), \quad \text{as} \quad n_k \neq n.
\]

So \(C = \bigcup_{i=1}^{4} C_i = \mathbb{Z}_n - \{0\}\) and \(\sum_{i=1}^{4} n_c^i = n\) for each \(c \in \mathbb{Z}_n - \{0\}\). Hence the theorem. \(\Box\)
4. Examples

Example 4.1. For $n = 5$; let $A_1 = \{0\}, B_1 = \{2, 3, 4, 5, 6, 7\}; A_2 = \{0, 2, 7\}, B_2 = \{1, 3, 4, 5, 6, 8\}; A_3 = \{0, 3, 6\}, B_3 = \{1, 2, 4, 5, 7, 8\}; A_4 = \{0, 4, 5\}, B_4 = \{1, 2, 3, 6, 7, 8\}$. Then $A_1 + B_1 = \{1, 2, 3, 4, 5, 6, 7\} A_2 + B_2 = \{1, 2, 3, 4\} A_3 + B_3 = \{1, 2, 3, 3, 4, 5\}$ and $A_4 + B_4 = \{1, 1, 2, 3, 6, 7, 7, 8\}$. These four H-partitions clearly satisfy the condition of the theorem and yield a set of four Williamson symmetric circulant matrices whose first rows are given by

\[
\begin{array}{cccccccc}
+1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
+1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
+1 & +1 & -1 & -1 & -1 & -1 & -1 & -1 \\
+1 & -1 & -1 & +1 & +1 & -1 & -1 & -1 \\
\end{array}
\]

Example 4.2. For $n = 9$; (i) $A_1 = \{0, 1, 8\}, B_1 = \{2, 3, 4, 5, 6, 7\}; A_2 = \{0, 2, 7\}, B_2 = \{1, 3, 4, 5, 6, 8\}; A_3 = \{0, 3, 6\}, B_3 = \{1, 2, 4, 5, 7, 8\}; A_4 = \{0, 4, 5\}, B_4 = \{1, 2, 3, 6, 7, 8\}$. Then $A_1 + B_1 = \{1, 2, 2, 3, 3, 4, 4, 4, 5, 5, 5, 5, 6, 6, 6, 7, 7, 8\}$ $A_2 + B_2 = \{1, 1, 1, 2, 3, 3, 4, 4, 5, 5, 6, 6, 6, 7, 7, 8, 8, 8\}$ $A_3 + B_3 = \{1, 1, 1, 2, 2, 2, 4, 4, 4, 5, 5, 5, 7, 7, 7, 8, 8, 8\}$ and $A_4 + B_4 = \{1, 1, 2, 2, 2, 3, 3, 4, 5, 6, 6, 6, 6, 7, 7, 7, 7, 7, 8\}$. These four H-partitions clearly satisfy the condition of the theorem and yield a set of four Williamson symmetric circulant matrices whose first rows are given by

\[
\begin{array}{cccccccccccc}
+1 & +1 & -1 & -1 & -1 & -1 & -1 & -1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \\
+1 & -1 & +1 & -1 & -1 & -1 & -1 & -1 & -1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \\
+1 & -1 & -1 & +1 & -1 & -1 & -1 & -1 & +1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
+1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
\end{array}
\]

as listed in [2]. Some other sets of such matrices are obtained by considering the partitions,

(ii) $A_1 = \{0, 1, 8\}, B_1 = \{2, 3, 4, 5, 6, 7\}; A_2 = \{0, 1, 3, 6, 8\}, B_2 = \{2, 4, 5, 7\}; A_3 = \{0, 2, 3, 6, 7\}, B_3 = \{1, 4, 5, 8\}; A_4 = \{0, 1, 3, 4, 5, 6, 8\}, B_4 = \{2, 7\}.$

(iii) $A_1 = \{0, 2, 7\}, B_1 = \{1, 3, 4, 5, 6, 8\}; A_2 = \{0, 2, 3, 6, 7\}, B_2 = \{1, 4, 5, 8\}; A_3 = \{0, 3, 4, 5, 6\}, B_3 = \{1, 2, 7, 8\}; A_4 = \{0, 1, 2, 3, 6, 7, 8\}, B_4 = \{4, 5\}.$

(iv) $A_1 = \{0, 4, 5\}, B_1 = \{1, 2, 3, 6, 7, 8\}; A_2 = \{0, 3, 4, 5, 6\}, B_2 = \{1, 2, 7, 8\}; A_3 = \{0, 1, 3, 6, 8\}, B_3 = \{2, 4, 5, 7\}; A_4 = \{0, 2, 3, 4, 5, 6, 7\}, B_4 = \{1, 8\}.$

The first row of the respective sets of Williamson matrices are,

(ii)
5. Possible size of partitions for Williamson matrices

**Theorem 5.1.** Let \((A_i, B_i), i = 1, 2, 3, 4\) be a set of H-partitions of \(\mathbb{Z}_n\), which gives rise to a set of Williamson matrices. Then \(\sum_{i=1}^{4} k_i(n - k_i) = n(n - 1)\), where \(k_i = |A_i|; i = 1, 2, 3, 4\).

**Proof.** Let \((A_i, B_i); i = 1, 2, 3, 4\) be a set of H-partitions of \(\mathbb{Z}_n\), which constructs a Hadamard matrix. Then \(\sum_{i=1}^{4} n_c^i = n\) for all \(c \in \mathbb{Z}_n - \{0\}\). Let \(k_i = |A_i|; i = 1, 2, 3, 4\).

Without loss of generality we can assume that \(0 \in A_i; i = 1, 2, 3, 4\). As \(A_i = -A_i; i = 1, 2, 3, 4\), \(k_i\) is an odd positive integer and consequently \(|B_i| = n - k_i\) is an even integer for all \(i = 1, 2, 3, 4\). Since \(A_i + B_i\) is a \(k_i \times (n - k_i)\) sub-matrix of the matrix corresponding to the composition table of \(\mathbb{Z}_n\), for \(i = 1, 2, 3, 4\); we have.

\[
\sum_{c \in \mathbb{Z}_n} n_c^i = k_i(n - k_i); i = 1, 2, 3, 4.
\]
\[ \Rightarrow \sum_{i=1}^{4} \left( \sum_{c \in \mathbb{Z}_n} n^i_c \right) = \sum_{i=1}^{4} k_i (n - k_i) \quad (13) \]

Again

\[ \sum_{i=1}^{4} \left( \sum_{c \in \mathbb{Z}_n} n^i_c \right) = \sum_{i=1}^{4} \left( \sum_{c \in \mathbb{Z}_n - \{0\}} n^i_c \right) \quad \text{as} \quad n^i_0 = 0; i = 1, 2, 3, 4 \]

\[ = \sum_{c \in \mathbb{Z}_n - \{0\}} \left( \sum_{i=1}^{4} n^i_c \right) \]

\[ \Rightarrow \sum_{i=1}^{4} \left( \sum_{c \in \mathbb{Z}_n} n^i_c \right) = \sum_{c \in \mathbb{Z}_n - \{0\}} n = n(n-1) \quad (14) \]

From (13) and (14) we have

\[ \sum_{i=1}^{4} k_i (n - k_i) = n(n-1). \]

So the possible size of \( A_i; \ i = 1, 2, 3, 4 \) are \( k_1, k_2, k_3 \) and \( k_4 \) respectively which is a set of odd integer solution of the equation

\[ w(n - w) + x(n - x) + y(n - y) + z(n - z) = n(n - 1) \]

**Theorem 5.2.** The equation

\[ w(n - w) + x(n - x) + y(n - y) + z(n - z) = n(n - 1) \]

has an integer solution if and only if there exists an integer solution of the equation

\[ X_1 + X_2 + X_3 + X_4 = n - 1 \]

in \( \{m(m-1)\}_{m=0}^{\infty} \).

**Proof.** Let \( \{k_1, k_2, k_3, k_4\} \) be an integer solution of the equation

\[ w(n - w) + x(n - x) + y(n - y) + z(n - z) = n(n - 1). \quad (15) \]
Thus
\[
\sum_{i=1}^{4} k_i(n-k_i) = n(n-1).
\]

Let \( X_i = (\frac{n-1}{2})(\frac{n+1}{2}) - k_i(n-k_i), i = 1, 2, 3, 4. \)

Since \( k_i + (n-k_i) = n; i = 1, 2, 3, 4 \), so \( (\frac{n-1}{2})(\frac{n+1}{2}) \geq k_i(n-k_i); i = 1, 2, 3, 4 \)
\( \Rightarrow X_i = (\frac{n-1}{2})(\frac{n+1}{2}) - k_i(n-k_i) \geq 0; i = 1, 2, 3, 4 \)

Then
\[
\sum_{i=1}^{4} X_i = \sum_{i=1}^{4} \{(\frac{n-1}{2})(\frac{n+1}{2}) - k_i(n-k_i)\}
\]
\[
= (n-1)(n+1) - \sum_{i=1}^{4} k_i(n-k_i)
\]
\[
= (n-1)(n+1) - n(n-1) \quad [from (15)]
\]
\[
= n - 1
\]

Now we have to show that \( X_i \in \{m(m-1)\}_{m=0}^{\infty} \) for \( i = 1, 2, 3, 4. \)

For \( i = 1, 2, 3, 4 \) we have
\[
X_i = (\frac{n-1}{2})(\frac{n+1}{2}) - k_i(n-k_i)
\]
\[
= (\frac{n-1}{2})(\frac{n+1}{2}) - k_i(\frac{n+1}{2}) + k_i(\frac{n+1}{2}) - k_i(n-k_i)
\]
\[
= (\frac{n+1}{2})(\frac{n-1}{2} - k_i) - k_i(\frac{n-1}{2} - k_i)
\]
\[
= (\frac{n+1}{2} - k_i)(\frac{n-1}{2} - k_i)
\]
\[
= m_i(m_i - 1) \quad [say \ m_i = \frac{n+1}{2} - k_i]
\]

If \( \frac{n+1}{2} > k_i \Rightarrow m_i > 0 \Rightarrow m_i(m_i - 1) \geq 0 \Rightarrow X_i \geq 0. \)

If \( \frac{n+1}{2} \leq k_i \Rightarrow m_i \leq 0 \Rightarrow m_i(m_i - 1) \geq 0 \Rightarrow X_i \geq 0. \)

Thus for \( i = 1, 2, 3, 4; \ X_i \in \{m(m-1)\}_{m=1}^{\infty}. \)

Conversely, let \( m_i(m_i - 1); i = 1, 2, 3, 4 \) be an integer solution of
\[
X_1 + X_2 + X_3 + X_4 = n - 1 \quad (16)
\]

Then \( \sum_{i=1}^{4} m_i(m_i - 1) = n - 1. \) We claim that for \( i = 1, 2, 3, 4; \ m_i \leq \frac{n-1}{2}. \) If
not, suppose for some \( i = 1, 2, 3, 4; m_i > \frac{n-1}{2} \Rightarrow m_i(m_i - 1) > \frac{n-1}{2} \frac{n+1}{2} \) for \( n \geq 3 \). For \( n = 1 \), \( X_1 = X_2 = X_3 = X_4 = 0 \) is a solution of (16) and the corresponding solution of (15) is \( w = x = y = z = 1 \).

Now consider \( k_i = \frac{n+1}{2} - m_i : i = 1, 2, 3, 4 \).

Then

\[
\sum_{i=1}^{4} k_i(n - k_i) = \sum_{i=1}^{4} \left( \frac{n+1}{2} - m_i \right) \left\{ n - \left( \frac{n+1}{2} - m_i \right) \right\} \\
= \sum_{i=1}^{4} \left( \frac{n+1}{2} - m_i \right) \left( \frac{n-1}{2} + m_i \right) \\
= \sum_{i=1}^{4} \left\{ \left( \frac{n+1}{2} \right) \left( \frac{n-1}{2} \right) + m_i \left( \frac{n+1}{2} - \frac{n-1}{2} \right) - m_i^2 \right\} \\
= 4 \left( \frac{n+1}{2} \right) \left( \frac{n-1}{2} \right) - \sum_{i=1}^{4} m_i(m_i - 1) \\
= (n+1)(n-1) - (n-1) \\
= n(n-1).
\]

So \( k_i(n - k_i); i = 1, 2, 3, 4 \) is a solution set of equation (15).

**Example.** For \( n = 31 \); the solutions of the equation

\[
X_1 + X_2 + X_3 + X_4 = n - 1,
\]

in \( \{m(m+1)\}_{m=0}^{\infty} \) are given by

(i) \((12, 12, 6, 0)\), (ii) \((12, 6, 6, 6)\), (iii) \((30, 0, 0, 0)\) and (iv)\((20, 6, 2, 2)\). Using theorem (5.2) the corresponding solutions of

\[
w(n - w) + x(n - x) + y(n - y) + z(n - z) = n(n - 1)
\]

are (a) \((19, 19, 13, 15)\), (b) \((19, 13, 13, 13)\), (c) \((21, 15, 15, 15)\) and (d) \((11, 13, 17, 17)\) \([\text{taking all odd solutions}]\) respectively. So possible size of part \( A_i \) of the \( H \)-partitions \( (A_i, B_i); i = 1, 2, 3, 4 \) are given by one of the solutions (a), (b), (c) and (d) only. Using these concepts the exhaustive search becomes quite easy as other sizes of \( H \)-partitions are disposed off.
Let us consider the solution (a) (19,19,13,15). By hit and trial we obtain

\[
A_1 = \{0, 1, 2, 4, 7, 10, 11, 12, 14, 15, 16, 17, 19, 20, 21, 24, 27, 29, 30\},
\]
\[
A_2 = \{0, 4, 5, 8, 9, 10, 11, 12, 14, 15, 16, 17, 19, 20, 21, 22, 26, 27\},
\]
\[
A_3 = \{0, 2, 6, 9, 12, 14, 15, 16, 17, 19, 22, 25, 29\}
\]

and
\[
A_4 = \{0, 2, 3, 4, 9, 10, 11, 13, 18, 20, 21, 22, 27, 28, 29\};
\]

such that the frequencies \(n_j^i; j = 1, 2, \ldots, 15; i = 1, 2, 3, 4\) are as follows:

\[
\begin{align*}
    n_j^1 &= \{8, 8, 6, 8, 7, 9, 10, 8, 8, 7, 7, 6, 7\}, \\
    n_j^2 &= \{6, 9, 8, 5, 7, 8, 8, 8, 8, 7, 10, 9, 8\}, \\
    n_j^3 &= \{10, 7, 6, 8, 9, 7, 8, 7, 9, 8, 8, 7, 7\}, \\
    n_j^4 &= \{7, 7, 11, 10, 8, 5, 8, 6, 9, 6, 9, 9\}.
\end{align*}
\]

Since \(\sum_{i=1}^{4} n_j^i = 31\) for \(j = 1, 2, \ldots, 15\), the conditions of the theorem (3.1) are satisfied by this set of four H-partitions and we have a set of four Williamson matrices giving rise to a Hadamard matrix of order \(4 \times 31\).

If we consider the solution (a) (19, 13, 13, 13). By hit and trial we obtain

\[
A_1 = \{0, 4, 5, 6, 8, 10, 11, 12, 13, 15, 16, 18, 19, 20, 21, 23, 25, 26, 27\},
\]
\[
A_2 = \{0, 1, 2, 3, 7, 9, 14, 17, 22, 24, 28, 29, 30\},
\]
\[
A_3 = \{0, 2, 3, 9, 11, 12, 15, 16, 19, 20, 22, 28, 29\}
\]

and
\[
A_4 = \{0, 2, 3, 9, 11, 12, 15, 16, 19, 20, 22, 28, 29\};
\]

such that the frequencies \(n_j^i; j = 1, 2, \ldots, 15; i = 1, 2, 3, 4\) are as follows:

\[
\begin{align*}
    n_j^1 &= \{8, 7, 9, 7, 6, 7, 8, 6, 10, 6, 8, 8, 9, 9, 6\}, \\
    n_j^2 &= \{7, 6, 8, 7, 8, 7, 7, 9, 7, 9, 9, 10, 8, 7\}, \\
    n_j^3 &= \{8, 9, 7, 8, 9, 8, 9, 6, 9, 7, 7, 6, 7, 9\}, \\
    n_j^4 &= \{8, 9, 7, 8, 9, 8, 9, 6, 9, 7, 7, 6, 7, 9\}.
\end{align*}
\]
Since $\sum_{i=1}^{4} n^i_j = 31$ for all $j = 1, 2, \ldots, 15$, the conditions of the theorem (3.1) are satisfied by this set of four $H$-partitions and we have a set of four Williamson matrices giving rise to a Hadamard matrix of order $4 \times 31$. Both of these are listed in [3].

**Remark.** It can be observed that the set of $H$-partitions which construct Williamson matrices also yields Supplementary Difference Sets [7, 8].

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**References**

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