FACTORS FOR $|\tilde{N}, p_n, \theta_n|_k$ SUMMABILITY OF FOURIER SERIES

BY

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Abstract

In the present paper, the author presents a generalization of some known results on the $|\tilde{N}, p_n|_k$ summability factors for the $|\tilde{N}, p_n, \theta_n|_k$ summability factors. Some new results have also been obtained.

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums $(s_n)$. We denote by $t_n$ the $n$-th $(C,1)$ mean of the sequence $(na_n)$. A series $\sum a_n$ is said to be summable $|C,1|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_n|^k < \infty. \quad (1)$$

Let $(p_n)$ be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_i = P_{-i} = 0, i \geq 1). \quad (2)$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_n s_v \quad (3)$$

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defines the sequence \((\sigma_n)\) of the Riesz mean or simply the \((\tilde{N}, p_n)\) mean of the sequence \((s_n)\), generated by the sequence of coefficients \((p_n)\) (see [3]). The series \(\sum a_n\) is said to be summable \(|\tilde{N}, p_n|_k\), \(k \geq 1\), if (see [1])

\[
\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} |\Delta \sigma_{n-1}|^k < \infty, \tag{4}
\]

where

\[
\Delta \sigma_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \quad n \geq 1. \tag{5}
\]

In the special case \(p_n = 1\) for all values of \(n\) \(|\tilde{N}, p_n|_k\) summability is the same as \(|C, 1|_k\) summability.

Let \((\theta_n)\) be any sequence of positive real constants. The series \(\sum a_n\) is said to be summable \(|\tilde{N}, p_n, \theta_n|_k\), \(k \geq 1\), if (see [8])

\[
\sum_{n=1}^{\infty} \theta_n^{k-1} |\Delta \sigma_{n-1}|^k < \infty. \tag{6}
\]

In the special case if we take \(\theta_n = \frac{p_n}{P_n}\), then \(|\tilde{N}, p_n, \theta_n|_k\) summability reduces to \(|\tilde{N}, p_n|_k\) summability. Also if we take \(\theta_n = n\) and \(p_n = 1\) for all values of \(n\), then we get \(|C, 1|_k\) summability. Furthermore if we take \(\theta_n = n\), then \(|\tilde{N}, p_n, \theta_n|_k\) summability reduces to \(|R, p_n|_k\) (see [3]) summability.

Let \(f(t)\) be a periodic function with period \(2\pi\) and integrable \((L)\) over \((-\pi, \pi)\). Without any loss of generality we may assume that the constant term in the Fourier series of \(f(t)\) is zero, so that

\[
\int_{-\pi}^{\pi} f(t) \, dt = 0 \tag{7}
\]

and

\[
f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t). \tag{8}
\]

We write

\[
\varphi(t) = \frac{1}{2} \{f(x + t) + f(x - t)\}, \quad \varphi_1(t) = \frac{1}{2} \int_{0}^{t} \varphi(u) \, du.
\]
2. Known Results

In [7] Mishra has proved two theorems for $|\tilde{N}, p_n|$ summability factors. Later on, Bor [2] has generalized these theorems for $|\tilde{N}, p_n|_k$ summability factors in the following forms.

**Theorem A.** Let $(p_n)$ be a sequence such that
\begin{align*}
P_n &= O(np_n) \quad (9) \\
P_n \Delta p_n &= O(p_n p_{n+1}) \quad (10)
\end{align*}

If $\varphi_1(t)$ is of bounded variation in $(0, \pi)$ and $(\lambda_n)$ is a sequence such that
\begin{align*}
\sum_{n=1}^{\infty} \frac{1}{n} |\lambda_n|^k &< \infty \quad (11) \\
\sum_{n=1}^{\infty} |\Delta \lambda_n| &< \infty, \quad (12)
\end{align*}

then the series $\sum A_n(t) \frac{P_n \lambda_n}{np_n}$ is summable $|\tilde{N}, p_n|_k$, $k \geq 1$.

**Theorem B.** If the sequences $(p_n)$ and $(\lambda_n)$ satisfy the conditions (9)–(12) of Theorem A and
\begin{equation}
B_n \equiv \sum_{v=1}^{n} v a_v = O(n), \quad (13)
\end{equation}

then the series $\sum a_n \frac{P_n \lambda_n}{np_n}$ is summable $|\tilde{N}, p_n|_k$, $k \geq 1$.

3. Main results.

The aim of this paper is to generalize Theorem A and Theorem B for $|\tilde{N}, p_n, \theta_n|_k$ summability methods.

Now we shall prove the following theorems.

**Theorem 1.** Let $(\frac{P_n \lambda_n}{np_n})$ be a non-increasing sequence. If all conditions
of Theorem A are satisfied with the condition (11) replaced by:

\[ \sum_{n=1}^{\infty} \theta_n^{k-1} n^{-k} \lambda_n^k < \infty, \quad (14) \]

then the series \( \sum A_n(t) \frac{P_n \lambda_n}{np_n} \) is summable \( |\check{N}, p_n, \theta_n|_k, k \geq 1 \).

**Theorem 2.** If the conditions (9)–(10) and (12)–(14) are satisfied and \( \frac{\theta_n p_n}{P_n} \) is a non-increasing sequence, then the series \( \sum a_n \frac{P_n \lambda_n}{np_n} \) is summable \( |\check{N}, p_n, \theta_n|_k, k \geq 1 \).

**Remark.** It should be noted that if we take \( \theta_n = \frac{P_n}{p_n} \) in Theorem 1 and Theorem 2, then we get Theorem A and Theorem B, respectively. In this case the condition \( \frac{\theta_n p_n}{P_n} \) which is a non-increasing sequence is automatically satisfied and condition (14) reduces to condition (11).

We need the following lemmas for the proof of our Theorems.

**Lemma 1** ([7]). If \( \varphi_1(t) \) is of bounded variation in \( (0, \pi) \), then

\[ \sum v A_v(x) = O(n) \quad \text{as} \quad n \to \infty. \]

**Lemma 2** ([2]). If the sequence \( (p_n) \) such that conditions (9) and (10) of Theorem A are satisfied, then

\[ \Delta \left\{ \frac{P_n}{p_n n^{2}} \right\} = O\left( \frac{1}{n^2} \right). \]

**4. Proof of Theorem 2.**

Let \( (T_n) \) denotes the \( (\check{N}, p_n) \) mean of the series \( \sum a_n P_n \lambda_n (np_n)^{-1} \). Then, by definition, we have

\[
T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum_{r=0}^{v} a_r P_r \lambda_r (r p_r)^{-1} = \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{v-1}) a_v P_v \lambda_v (v p_v)^{-1}.
\]
Then, for $n \geq 1$, we have

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_v a_v P_v \lambda_v (v^2 p_v)^{-1}.$$ 

By Abel’s transformation, we have

$$T_n - T_{n-1} = B_n \lambda_n n^{-2} - p_n (P_n P_{n-1})^{-1} \sum_{v=1}^{n-1} P_v P_v B_v \lambda_v (v^2 p_v)^{-1}$$

$$+ p_n (P_n P_{n-1})^{-1} \sum_{v=1}^{n-1} P_v P_v \Delta \lambda_v B_v (v^2 p_v)^{-1}$$

$$+ p_n (P_n P_{n-1})^{-1} \sum_{v=1}^{n-1} P_v B_v \lambda_{v+1} \Delta \left\{ \frac{P_v}{v^2 p_v} \right\}$$

$$= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \quad \text{say.}$$

To prove the theorem, by Minkowski’s inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \theta_n^{k-1}|T_{n,r}|^k < \infty, \quad \text{for} \quad r = 1, 2, 3, 4.$$ 

Firstly, we have that

$$\sum_{n=1}^{m} \theta_n^{k-1}|T_{n,1}|^k = \sum_{n=1}^{m} \theta_n^{k-1} |\lambda_n|^k |B_n|^{k-2k}$$

$$= O(1) \sum_{n=1}^{m} \theta_n^{k-1} |\lambda_n|^k n^{k-2k}$$

$$= O(1) \sum_{n=1}^{m} \theta_n^{k-1} n^{k-2k} |\lambda_n|^k$$

$$= O(1) \quad \text{as} \quad m \to \infty,$$

by (13) and (14).

Now, applying Hölder’s inequality, we have that

$$\sum_{n=2}^{m+1} \theta_n^{k-1}|T_{n,2}|^k = \sum_{n=2}^{m+1} \theta_n^{k-1} \left( \frac{p_n}{P_n} \right)^k \frac{1}{P_{n-1}} \left| \sum_{v=1}^{n-1} \frac{p_v P_v B_v \lambda_v}{v^2 p_v} \right|^k.$$
\[
\begin{align*}
\leq & \sum_{n=2}^{m+1} \theta_n^{k-1} \left( \frac{p_n}{P_n} \right)^k \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \left\{ \frac{P_v |B_v||\lambda_v|}{v^2 p_v} \right\}^k \\
& \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1} \\
= & O(1) \sum_{v=1}^{m} p_v |\lambda_v| \left\{ \frac{p_v}{p_v} \right\}^k v^{k v - 2k} \sum_{n=v+1}^{m+1} \left( \frac{\theta_n p_n}{P_n} \right)^{k-1} \frac{p_n}{P_n P_{n-1}} \\
= & O(1) \sum_{v=1}^{m} |\lambda_v| \left\{ \frac{p_v}{p_v} \right\}^{k-1} v^{-k} \left( \frac{\theta_v p_v}{P_v} \right)^{k-1} \\
= & O(1) \sum_{v=1}^{m} \theta_v^{k-1} v^{-k} |\lambda_v|^k = O(1) \quad \text{as} \quad m \to \infty,
\end{align*}
\]

by (13) and (14). On the other hand, since
\[
\sum_{v=1}^{n-1} P_v |\Delta \lambda_v| \leq P_{n-1} \sum_{v=1}^{n-1} |\Delta \lambda_v| \Rightarrow \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v |\Delta \lambda_v| \leq \sum_{v=1}^{n-1} |\Delta \lambda_v| = O(1),
\]

by (12), we have that
\[
\begin{align*}
\sum_{n=2}^{m+1} \theta_n^{k-1} & |T_{n,3}|^k \\
\leq & \sum_{n=2}^{m+1} \theta_n^{k-1} \left( \frac{p_n}{P_n} \right)^k \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v P_v B_v |\Delta \lambda_v|^k \\
\leq & \sum_{n=2}^{m+1} \theta_n^{k-1} \left( \frac{p_n}{P_n} \right)^k \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v |\Delta \lambda_v| \left\{ \frac{P_v |B_v|}{v^2 p_v} \right\}^k \\
& \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v |\Delta \lambda_v| \right\}^{k-1} \\
= & O(1) \sum_{v=1}^{m} |B_v|^k v^{-2k} \left\{ \frac{P_v}{p_v} \right\}^k P_v |\Delta \lambda_v| \sum_{n=v+1}^{m+1} \left( \frac{\theta_n p_n}{P_n} \right)^{k-1} \\
& \times \frac{p_n}{P_n P_{n-1}} \\
= & O(1) \sum_{v=1}^{m} v^k v^{-2k} v^{k} |\Delta \lambda_v| \left( \frac{\theta_v p_v}{P_v} \right)^{k-1}
\end{align*}
\]
\[ O(1) \left( \frac{\theta_1 p_1}{P_1} \right)^{k-1} \sum_{v=1}^{m} |\Delta \lambda_v| \]

\[ = O(1) \sum_{v=1}^{m} |\Delta \lambda_v| = O(1) \quad \text{as} \quad m \to \infty, \]

in view of (9), (12) and (13).

Finally, using the fact that \( \Delta \left\{ \frac{P_r}{v^2 p_v} \right\} = O\left( \frac{1}{v} \right) \) by Lemma 2, we get

\[
\sum_{n=2}^{m+1} \theta_{n}^{k-1} |T_{n,4}|^k \leq \sum_{n=2}^{m+1} \theta_{n}^{k-1} \left( \frac{p_n}{P_n} \right)^k \frac{1}{p_{n-1}^{k-1}} \left\{ \sum_{v=1}^{n-1} p_v |B_v| \lambda_{v+1} \Delta \left\{ \frac{P_v}{v^2 p_v} \right\} \right\}^k
\]

\[ = O(1) \left( \frac{p_n}{P_n} \right)^k \frac{1}{p_{n-1}^{k-1}} \left\{ \sum_{v=1}^{n-1} p_v |B_v| \lambda_{v+1} \right\}^k
\]

\[ = O(1) \left( \frac{p_n}{P_n} \right)^k \frac{1}{p_{n-1}^{k-1}} \left( \frac{P_v}{p_v} \right)^k p_v |\lambda_{v+1}| v^{-2k}
\]

\[ \times |B_v|^k \left\{ \sum_{v=1}^{n-1} p_v \right\}^{k-1}
\]

\[ = O(1) \left( \frac{p_v}{P_v} \right)^k p_v |\lambda_{v+1}| v^{-2k} \sum_{n=v+1}^{m+1} \left( \frac{\theta_n p_n}{P_n} \right)^{k-1}
\]

\[ \times \frac{p_n}{P_n p_{n-1}}
\]

\[ = O(1) \left( \frac{p_v}{P_v} \right)^k v^{-k} |\lambda_{v+1}|^{k-1} \left( \frac{\theta_v p_v}{P_v} \right)^{k-1}
\]

\[ = O(1) \sum_{v=1}^{m} \theta_v^{k-1} v^{-k} |\lambda_{v+1}|^{k} = O(1) \quad \text{as} \quad m \to \infty, \]

by (13) and (14). Therefore, we get that

\[ \sum_{n=1}^{m} \theta_{n}^{k-1} |T_{n,r}|^k = O(1) \quad \text{as} \quad m \to \infty, \quad \text{for} \quad r = 1, 2, 3, 4. \]

This completes the proof of Theorem 2.

**Proof of Theorem 1.** Theorem 1 is a direct consequence of Theorem 2 and Lemma 1. If we take \( p_n = 1 \) and \( \theta_n = n \) in Theorem 1 and Theorem...
2, then we get the following corollaries. It should be noted that, in this case condition (14) reduces to condition (11).

**Corollary 1.** If \( \varphi_1(t) \) is of bounded variation in \((0, \pi)\) and \((\lambda_n)\) is a sequence such that conditions (11) and (12) are satisfied, then the series
\[
\sum A_n(t)\lambda_n, \text{ at } t = x \text{ is summable } |C, 1|_k, \ k \geq 1.
\]

**Corollary 2.** If the conditions (11)–(13) are satisfied, then the series
\[
\sum a_n\lambda_n \text{ is summable } |C, 1|_k, \ k \geq 1.
\]

**References**


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