COHOMOGENEITY TWO ACTIONS ON $\mathbb{R}^m, m \geq 3$

BY

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Abstract

We suppose that a connected and closed Lie group $G$ of isometries of $\mathbb{R}^m, m \geq 3$, acts by cohomogeneity two on $\mathbb{R}^m$. Then we show that under some conditions, the orbit space is homeomorphic to $\mathbb{R}^2$ or $[0, +\infty) \times \mathbb{R}$.

1. Introduction

Let $G$ be a connected and closed Lie group of isometries of a Riemannian manifold $M$. For each point $x \in M$, we denote the orbit containing $x$ by:

$$G(x) = \{gx : g \in G\}.$$  

We say that $G$ acts by “Cohomogeneity $K$” on $M$, if

$$\dim M = K + \max\{\dim G(x) : x \in M\}.$$  

If $K = 0$, then for each point $x \in M$, we have $M = G(x)$ and $M$ is called homogeneous $G$-manifold. Homogeneous and cohomogeneity one manifolds are studied by several authors (see [1], [2], [7], [10], [11]). Study of cohomogeneity two Riemannian manifolds is still wide open. In [3] the authors studied cohomogeneity two Riemannian manifolds from a algebraic view point. In [8] it is considered that $M$ is flat and $G$ has fixed point in $M$. Then the orbits and orbit space are characterized. In this paper we consider cohomogeneity two actions on $\mathbb{R}^m, m \geq 3$. In Theorem 3.6 we suppose that $G$ is a compact
connected subgroup of $\text{Isom}(R^m)$, which acts by cohomogeneity two on $R^m$. Then we show that the orbit space is homeomorphic to $[0, +\infty) \times R$. In Theorem 3.8 we suppose that $G$(compact or noncompact) has an irreducible orbit. Then we show that the orbit space is homeomorphic to $[0, +\infty) \times R$ or $R^2$.

2. Preliminaries

In this paper, when two spaces $X$ and $Y$ are homeomorphic we denote this by $X \sim Y$. Now, we mention some facts which we will use in the sequel. Let $G$ be a connected and closed Lie subgroup of isometries of $M$. We denote by $\overline{M}$ the set of orbits of this action and we equip $\overline{M}$ with the quotient topology relative to the canonical projection $M \rightarrow \overline{M}, x \rightarrow G(x)$.

**Definition 2.1.** Let $G$ and $H$ be closed and connected subgroups of $\text{Isom}(M)$. We say that $G$ and $H$ are orbit-equivalent on $M$, if the set of orbits of $G$-action on $M$ is equal to the set of orbits of $H$-action on $M$.

$$\{G(x) : x \in M\} = \{H(x) : x \in M\}$$

The following fact is clear.

**Fact 2.2.** If $G$ and $H$ are orbit-equivalent on $M$, then $\overline{M}_G = \overline{M}_H$.

**Fact 2.3.** Let $\widetilde{M}$ and $\widetilde{G}$ be the universal covering manifolds of $M$ and $G$, with covering maps $\pi : \widetilde{M} \rightarrow M$ and $\kappa : \widetilde{G} \rightarrow G$. It is well known (see [4] pages 62-63) that $\widetilde{G}$ acts on $\widetilde{M}$, such that for each $\tilde{x} \in \widetilde{M}$ and $\tilde{g} \in \widetilde{G}$ we have:

$$\pi(\tilde{g}\tilde{x}) = \kappa(\tilde{g})\pi(\tilde{x})$$

If $M$ is simply connected then $G$ and $\widetilde{G}$ both act on $M$ orbit equivalently and the map $\kappa : \widetilde{G} \rightarrow G$ is a representation of $\widetilde{G}$ as isometries of $M$ (the action of $\widetilde{G}$ on $M$ may be not effective).

**Definition 2.4.** Let $G$ be a connected and closed subgroup of isometries of $R^m$. In Fact 2.3 if we let $M = R^m$ then we have $\widetilde{M} = R^m$ and $\pi$ is the
identity map. So for each $\tilde{x} \in R^m$ and $\tilde{g} \in \tilde{G}$ we have

$$\tilde{g}\tilde{x} = \kappa(\tilde{g})\tilde{x}$$

The covering map $\kappa : \tilde{G} \to G$ is a representation of $\tilde{G}$ on $G \subset Isom(R^m)$. This representation of $\tilde{G}$ is called "induced representation".

By Fact 2.3 we have the following fact.

**Fact 2.5.** If $G$ is a closed and connected Lie subgroup of isometries of $R^m$, then the group $\tilde{G}$ the universal covering group of $G$ acts on $R^m$ by induced representation, orbit-equivalently to $G$, and we have:

$$\frac{R^m}{G} = \frac{R^m}{\tilde{G}}.$$

**Fact 2.6.** (See [2], [7] and [10]) Let $G$ act by cohomogeneity one on $M$, then

(a) The orbit space $\frac{M}{G}$ is homeomorphic to one of the following spaces

$$R; [0, +\infty); S^1; [-1, 1].$$

(b) If $M$ is simply connected, then $\frac{M}{G} \sim S^1$.

(c) If $M$ is compact, then $\frac{M}{G} \sim S^1$ or $\frac{M}{G} \sim [-1, 1]$.

The isometry group of $R^n$ is in the form $O(n) \times R^n$, where the action of $(A, b) \in O(n) \times R^n$ on $R^n$ is as follows:

$$(A, b)(x) = A(x) + b.$$

The isometry $(I, b)$ is called an ordinary translation.

$$(I, b)(x) = x + b.$$

**Fact 2.7.** If $R^n$ is of cohomogeneity one under the action of a closed and connected Lie subgroup $G$ of isometries of $R^n$, then

(a) $\frac{R^n}{G} \sim R$ or $\frac{R^n}{G} \sim [0, +\infty)$.

(b) If $G$ contains ordinary translations only, then $\frac{R^n}{G} \sim R$. 
Proof. (a). By Theorem 2.8 in [7], $\frac{\mathbb{R}^n}{G} \sim [-1,1]$ and by Fact 2.6(b), we have $\frac{\mathbb{R}^n}{G} \sim S^1$. Therefore $\frac{\mathbb{R}^n}{G} \sim R$ or $\frac{\mathbb{R}^n}{G} \sim [0, +\infty)$.

(b) If $G$ contains ordinary translations only, then for each two points $x,y \in \mathbb{R}^n$ we have

$$G(x) = \{x + b : b \in G\}, \quad G(y) = \{y + b : b \in G\}.$$ 

So all orbits are diffeomorphic to each other and there is not any singular orbit (see [7] proof of Theorem 3.1). Thus by part (a), we have $\frac{\mathbb{R}^n}{G} \sim R$. □

Definition 2.8. Let $M$ be a submanifold of $\mathbb{R}^m$, we say that $M$ is reducible, if $M$ is isometric to $M_1 \times M_2$, where $M_1, M_2$ are submanifolds of $\mathbb{R}^m$ and $\dim M_i \geq 1$.

3. Results

Before stating our results we give a definition and lemma in general topology.

Definition 3.1. Let $I = [0, +\infty)$, $X = \bigcup_{t \in I} X_t$, where $X$ is a topological space and for each $t$, $X_t$ is a subspace of $X$ and the union is disjoint. We say that $X$ is a continuous motion of $X_1$ on $I$, if there exist a continuous map $\psi : X_1 \times I \rightarrow X$ such that

1. $\psi(x, t) \in X_t$.
2. $\psi(x, 1) = x$.
3. The collection $B$ containing all of the sets in the form $\psi(U \times (a,b))$ and $\psi(X_1 \times [0,b))$ is a basis for the topology of $X$, (where $(a,b) \subset I$ and $U$ is open in $X_1$).

The map $\psi$ is called motion map.

Example 3.2. Let $X = \mathbb{R}^2$, $X_t = S^1(t) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = t^2\}$ and let $\psi : S^1(1) \times I \rightarrow \mathbb{R}^2$, $\psi(x, t) = tx$. It is easy to see that $X$ is a continuous motion of $X_1$.

Lemma 3.3. Let $X = \bigcup_t X_t$, $Y = \bigcup_t Y_t$ be two spaces which are continuous motions of $X_1, Y_1$ under the motions $\psi : X_1 \times I \rightarrow X$ and
\( \phi : Y_1 \times I \rightarrow Y. \) Also let for each \( t \) in \( I \), there is a homeomorphism \( F_t : X_t \rightarrow Y_t \), such that

\( \phi_t \circ \psi_t = \phi_t \circ F_1(\ast) \)

where \( \psi_t(x) = \psi(x,t), \phi_t(x) = \phi(x,t) \). Then \( X \) is homeomorphic to \( Y \).

**Proof.** Consider the map \( F \) as:

\[ F : X \rightarrow Y, F(x) = F_1(x), x \in X_t. \]

By definition, the collection \( B = \{ \phi(V \times (a,b)), \phi(Y_1 \times [0,b]) : V \text{ open in } Y_1, (a,b) \subset I \} \) is a basis for topology of \( Y \). \( F_1 \) is a homeomorphism. So if \( V \) is open in \( Y_1 \) then \( U = F_1^{-1}(V) \) is open in \( X_1 \). By using (*), we have:

\[
F^{-1}\{\phi(V \times (a,b))\} = \bigcup_{t \in (a,b)} F_t^{-1}\{\phi(x,t) : x \in V\} = \bigcup_t F_t^{-1}\{\phi_t(x) : x \in V\}
\]

\[
= \bigcup_t \{\psi_t \circ F_1^{-1}(x) : x \in V\} = \bigcup_t \{\psi_t(y) : y \in U\}
\]

\[
= \psi(U \times (a,b)).
\]

In similar way we can show that:

\[ F^{-1}(\phi(Y_1 \times [0,b])) = \psi(X_1 \times [0,b]). \]

So for each open set \( W \) in \( Y \), \( F^{-1}(W) \) is open in \( X \). This means that \( F \) is continuous. In the similar way we can show that \( F^{-1} \) is continuous. Therefore \( F \) is a homeomorphism between \( X \) and \( Y \). \( \square \)

**Theorem 3.4.** ([5, p.56]) Let \( M = G(x) \) be a homogeneous irreducible submanifold of \( \mathbb{R}^n \), where \( G \) is a connected Lie subgroup of isometries of \( \mathbb{R}^n \). Then the universal covering group \( \tilde{G} \) of \( G \) is isomorphic to the direct product \( K \times \mathbb{R}^d \), where \( K \) is a simply connected Lie group. Moreover, the induced representation of \( \tilde{G} \) is equivalent to \( P_1 \bigoplus P_2 \) where \( P_1 \) is a representation of \( \tilde{G} \) in to \( SO(d) \) and \( P_2 \) is linear map from \( \mathbb{R}^d \) to \( \mathbb{R}^e \), \( n = d + e \) regarding \( \mathbb{R}^e \) as ordinary translations.

From Theorem 3.4 and its proof (in [5] pages 56, 57) we get the following corollary.
Corollary 3.5. If $M = G(x)$ is a homogeneous irreducible submanifold of $\mathbb{R}^n$, then $\tilde{G}$, the universal covering group of $G$, is orbit equivalent to a subgroup $H$ of the group $\{(A,b) : A \in SO(d), b \in \mathbb{R}^e\}$, where $H$ acts on $\mathbb{R}^n$, as follows

$$(A,b)(x,y) = (Ax, y + b); \ (x,y) \in \mathbb{R}^d \times \mathbb{R}^e = \mathbb{R}^n.$$ 

Theorem 3.6. If $G \subset ISO(\mathbb{R}^m)$ is compact and connected and acts by cohomogeneity two on $\mathbb{R}^m, m \geq 3$, then

$$\frac{\mathbb{R}^m}{G} \sim [0, +\infty) \times \mathbb{R}.$$ 

Proof. Since $G$ is compact, by Cartan’s theorem (see [6] vol II page 111) it has at least one fixed point in $\mathbb{R}^m$. Without loss of generality, we assume that the origin is a fixed point of $G$. Let $S^{m-1}(r)$ be the standard sphere of radius $r$ in $\mathbb{R}^m$.

$$S^{m-1}(r) = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : \sum_{i=1}^{m} x_i^2 = r^2\}.$$

Since each $g \in G$ fixes the origin of $\mathbb{R}^m$ invariant, for any $x \in S^{m-1}(r)$ we have $g(x) \in S^{m-1}(r)$. So we can consider $G$ as a subgroup of isometries of $S^{m-1}(r)$ (i.e. $G \subset O(m)$). Let $r_2 > r_1 > 0$ and consider the following map

$$\begin{cases} 
\phi_{r_1 r_2} : S^{m-1}(r_1) \to S^{m-1}(r_2), \\
\phi_{r_1 r_2}(x) = \frac{r_2}{r_1} x.
\end{cases}$$

Each $g \in G$ is a linear map on $\mathbb{R}^m$. So we have:

$$\phi_{r_1 r_2}(gx) = \frac{r_2}{r_1} (gx) = g(\frac{r_2}{r_1} x) = g\phi_{r_1 r_2}(x).$$

Therefore $\phi_{r_1 r_2}$ maps each orbit of the $G$-action on $S^{m-1}(r_1)$ diffeomorphically on to an orbit of $G$-action on $S^{m-1}(r_2)$. So topologically, the orbit foliation of $S^{m-1}(r_1)$ is alike the orbit foliation of $S^{m-1}(r_2)$. Since $\mathbb{R}^m$ is of cohomogeneity two under the action of $G$, then for each $r > 0, S^{m-1}(r)$ is of cohomogeneity one. Consider the sphere $S^{m-1}(1)$. By Fact 2.6(b,c), $\frac{S^{m-1}(1)}{G}$. 

is homeomorphic to $[-1,1]$. Let $P$ be this homeomorphism.

$$P : \frac{S^{m-1}(1)}{G} \to [-1,1].$$

We have $\mathbb{R}^m = \bigcup_{t \in I} S^{m-1}(t)$, where $I = [0, +\infty)$. So $\frac{\mathbb{R}^m}{G} = \bigcup_{t} \frac{S^{m-1}(t)}{G}$. Let $X_t = \frac{S^{m-1}(t)}{G}$, $X = \frac{\mathbb{R}^m}{G}$, it is easy to see that $X$ is a continuous motion of $X_1$ under the motion map $\psi$ defined by:

$$\psi : X_1 \times I \longrightarrow X; \psi(G(x), t) = G(tx), x \in S^{m-1}(1).$$

Let $Y$ be the subset of $\mathbb{R}^2$ defined by:

$$Y = \bigcup_{t \in I} \{t\} \times [-t, t].$$

and let

$$Y_t = \{t\} \times [-t, t], t \in I.$$

$Y$ is a continuous motion of $Y_1 = \{1\} \times [-1, 1]$ under the map $\phi$ defined by:

$$\phi : Y_1 \times I \longrightarrow Y, \phi((1, a), t) = (t, ta).$$

Now for each $t$ in $I$ define the map $F_t : X_t \longrightarrow Y_t$ as follows

$$\begin{cases} F_t(G(x)) = (t, tP(G(\frac{x}{|x|}))), & |t| \neq 0, \\ F_0(o) = (0, 0), & |t| = 0. \end{cases}$$

Note that $X_0 = \{o\}, Y_0 = \{(0, 0)\}$. For each $t$ in $I$, $F_t$ is homeomorphism and the conditions of Lemma 3.3 are valid. Thus $X$ is homeomorphic to $Y$. But easily we can show that $Y$ is homeomorphic to $[0, +\infty) \times R$. Therefore $X$ is homeomorphic to $[0, +\infty) \times R$.

**Lemma 3.7.** Let $H$ be a closed and connected subgroup of $SO(d) \times \mathbb{R}^e$ which acts by cohomogeneity two on $\mathbb{R}^d \times \mathbb{R}^e = \mathbb{R}^m$ and let

$$S = \{A : (A, b) \in H \text{ for some } b \in \mathbb{R}^e\},$$

$$T = \{b : (A, b) \in H \text{ for some } A \in SO(d)\}.$$

Then
(1) One of the following is true.
(a) The cohomogeneity of $S$-action on $R^d$ is 1 and the cohomogeneity of $T$-action on $R^e$ is 1 or 0.
(b) The cohomogeneity of $S$-action on $R^d$ is 2 and the cohomogeneity of $T$-action on $R^e$ is 0.

(2) For $r > 0$, let $M_r = S^{d-1}(r) \times R^e \subseteq R^d \times R^e$, where $S^{d-1}(r)$ is the standard sphere in $R^d$ with radius $r$. Then for each $r > 0$, $H$ acts by cohomogeneity one on $M_r$ and for each $r_1, r_2 > 0$ we have $\frac{M_{r_1}}{H} \sim \frac{M_{r_2}}{H}$.

(3) In (2), if for one $r > 0$, $\frac{M_r}{H}$ is compact, then $\frac{M_0}{H}$ is a one point space.

(4) $\frac{R^m}{H}$ is homeomorphic to $[0, +\infty) \times R$ or $R^2$.

Proof. (1) Since $H \subset S \times T$, we have:

$$2 = \text{cohomogeneity of } H \text{ action on } R^m \geq \text{cohomogeneity of }$$

$$+S\text{-action on } R^d \text{ cohomogeneity of } T\text{-action on } R^e.$$ 

Since $S$ is compact, it has fixed point in $R^d$. Thus the cohomogeneity of $S$-action on $R^d$ is $\geq 1$. These yield to (a) or (b).

(2) Consider $(x, y) \in M_r, x \in S^{d-1}(r), y \in R^e$, we have:

$$H(x, y) \subseteq (S \times T)(x, y) = S(x) \times T(y) \subseteq S^{d-1}(r) \times R^e = M_r.$$ 

So $H$ maps $M_r$ on to itself and we can consider $H$ as a subgroup of isometries of $M_r$. For $r_1, r_2 > 0$, the map $\varphi_{r_1 r_2} : M_{r_1} \to M_{r_2}; (x, y) \to (\frac{r_2 x}{r_1}, y)$ induces a homeomorphism between $\frac{M_{r_1}}{H}$ and $\frac{M_{r_2}}{H}$. Since $\text{dim}M_r = m - 1$ and the action of $H$ on $R^m$ is of cohomogeneity two, the action of $H$ on $M_r$ is of cohomogeneity one.

(3) Consider the map: $\phi_r : M_r \to M_0$ defined by $:\phi_r(x, y) = y$. $\phi_r$ induces a continuous and on to map: $\bar{\phi}_r : \frac{M_r}{H} \to \frac{M_0}{H}$. So $\frac{M_0}{H}$ must be compact. But it is easy to see that $\frac{M_0}{H} = \frac{R^e}{R}$. By part (1) of Lemma and Fact 2.7, we have $\frac{R^e}{R} = \{0\}$ or $R$. Since $\frac{M_0}{H} \sim \frac{R^e}{R}$ is compact, we get that $\frac{M_0}{H} \sim \frac{R^e}{R} = \{0\}$.

(4) We have $R^m = \bigcup_{t \in I} M_t$, where $I = [0, +\infty)$. So 

$$\frac{R^m}{H} = \bigcup_t \frac{M_t}{H}.$$
Let

\[ X = \frac{R^m}{H}, X_t = \frac{M_t}{H}. \]

\( X \) is a continuous motion of \( X_1 \) under the motion map \( \psi \) defined by

\[ \psi : X_1 \times I \longrightarrow X, \psi(H(x,y),t) = H(tx,y); (x,y) \in M_1 = S^{d-1}(1) \times R^e \]

By Fact 2.6(a) and part (2) of Lemma, for all \( r > 0 \), \( \frac{M_r}{H} \) is homomorphic to one of the following spaces.

(I) \( S^1(r) \) (II) \([-r,r]\) (III) \([0, +\infty)\) (IV) \( R \).

We study each case separately

(I) \( \frac{M_r}{H} \sim S^1(r), r > 0 \).

Let

\[ Y = R^2, Y_t = S^1(t), t \in [0, +\infty) \]

\( Y \) is a continuous motion of \( Y_1 \) under the map:

\[ \phi : Y_1 \times I \longrightarrow Y, \phi(a,t) = ta. \]

Let \( P \) be the homeomorphism between \( X_1 = \frac{M_r}{H} \) and \( Y_1 = S^1(1) \). For each \( t \) in \( I \), define the map \( F_t : X_t \longrightarrow Y_t \) as follows:

\[
\begin{cases}
F_t(H(x,y)) = tP(H(\frac{x}{|x|}, y)), & t \neq 0, \\
F_0(o) = (0,0), & t = 0.
\end{cases}
\]

Note that \( Y_0 = (0,0) \) and by part (3) of Lemma, we have \( X_0 = \{o\} \). For each \( t \in I \), \( F_t \) is homeomorphism and the conditions of Lemma 3.3 are valid. So \( X \) is homeomorphic to \( Y = R^2 \).

(II) \( \frac{M_r}{H} = [-r,r], r > 0 \).

In this case we let

\[ Y = \bigcup_t Y_t \]

where

\[ Y_t = t \times [-t,t], \]
\[ \phi : Y_1 \times I \rightarrow Y, \quad \phi((1, a), t) = (t, ta). \]

As like as the proof of Theorem 3.6, we can show that \( X \) is homeomorphic to \( Y \). Since \( Y \) is homeomorphic to \([0, \infty) \times R\), we get that \( X \) is homeomorphic to \([0, \infty) \times R\).

(III) \( \frac{M_r}{H} \sim [0, +\infty), \quad r > 0 \).

We show that this case can not occur. Consider the continuous and onto map

\[
\begin{cases}
\phi_r : M_r \rightarrow R^e \\
\phi_r(x, y) = y
\end{cases}
\]

\( \phi_r \) induces continuous and onto map \( \overline{\phi_r} \) between orbit spaces

\[
\phi_r : \frac{M_r}{H} \sim [0, +\infty) \rightarrow \frac{R^e}{T}.
\]

By part (1) of Lemma and Fact 2.7, \( \frac{R^e}{T} \) is homeomorphic to \( \{0\} \) or \( R \). If \( \frac{R^e}{T} \sim R \) then \( \overline{\phi} \) is a continuous and onto map as follows:

\[
\overline{\phi_r} : [0, +\infty) \rightarrow R.
\]

So the following map is continuous and onto

\[
\overline{\phi_r} : (0, +\infty) \rightarrow R - \{\overline{\phi_r}(0)\},
\]

which is a contradiction (because \( R - \{\overline{\phi_r}(0)\} \) is not connected.)

If \( \frac{R^e}{T} = 0 \), then \( T \) acts transitively on \( R^e \). So for each \((x, y) \in M_r\) there exists \((A, b) \in H\) such that \((A, b)(x, y) = (x_1, 0)\) for some \( x_1 \in S^{d-1}(r) \). Thus each \( H \)-orbit of \( M_r \) intersects the set \( S^{d-1}(r) \times \{0\} \subset S^{d-1}(r) \times R^e = M_r \). Let \( \kappa : M_r \rightarrow \frac{M_r}{H} \) be the projection on the orbit space and consider the map \( \eta : M_r \rightarrow S^{d-1}(r) \times \{0\}, \eta(x, y) = (x, 0) \) and let \( \kappa_1 \) be the restriction of \( \kappa \) on \( S^{d-1}(r) \times \{0\} \). Easily we see that the following diagram is commutative

\[
\begin{cases}
\eta : M_r \rightarrow S^{d-1} \times \{0\}
\end{cases}
\]

\[
\kappa \downarrow \quad \overline{\kappa_1}
\]

\[
\frac{M_r}{H}
\]
Since $S^{d-1}(r) \times \{0\}$ is compact, $\frac{M_r}{H}$ must be compact, which is a contradiction. Therefore the case III cannot occur.

(IV) $\frac{M_r}{H} \sim R, \ r > 0$.

If $\frac{R^e}{H} = 0$. As like as the case III we get a contradiction. Let $\frac{R^e}{H} = R$, we have:

$$R^m = (\{0\} \times R_e) \cup (\cup_{r>0} M_r)$$

and $\frac{\{0\} \times R_e}{H} = \frac{R^e}{H} = R$. Let

$$Y = [0, +\infty) \times R, Y_t = \{t\} \times R$$

$Y$ is a continuous motion of $Y_1$, by the map $\phi : Y_1 \times I \longrightarrow Y$ defined by

$$\phi((1, a), t) = (t, a).$$

As like as before by suitable choice of the maps $F_t : X_t \longrightarrow Y_t$, we can show that $X$ is homeomorphic to $Y = [0, +\infty) \times R$. □

**Theorem 3.8.** Let $R^m, m > 3$, be of cohomogeneity two, under the action of a connected and closed Lie subgroup $G$ of $\text{Isom}(R^m)$, and suppose that there exists an irreducible orbit $G(x)$ for some $x$ in $R^m$, then $\frac{R^m}{G}$ is homeomorphic to one of the following spaces:

$$[0, +\infty) \times R; \ R^2$$

**Proof.** Let $G(x)$ be an irreducible orbit of this action. By Corollary 3.5, $\hat{G}$ the universal covering Lie group of $G$ acts on $R^m$, orbit-equivalent to a subgroup of the group $\{(A, b) : A \in SO(d), b \in R^e\} = SO(d) \times R^e$, $d + e = m$, which we denote it by $H$. By Fact 2.5 and Corollary 3.5, we get that:

$$\frac{R^m}{G} \sim \frac{R^m}{H}$$

So we get the result by Lemma 3.7(4). □

**References**


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