UNIQUENESS THEOREMS FOR PERIODIC SOLUTIONS
OF CERTAIN FOURTH AND FIFTH ORDER
DIFFERENTIAL SYSTEMS

BY

ERCAN TUNÇ

Abstract

In this paper, we establish sufficient conditions which guarantee the existence at the most one $\omega$-periodic solution for certain two class of fourth and fifth order differential equations. Our results extend some well-known results carried out in the relevant literature.

1. Introduction and Statement of the Result

We consider fourth and fifth order nonlinear vector differential equations

$$X^{(4)} + A_1 \ddot{X} + F(\dot{X}) + A_3 \dot{X} + G(X) = P_1(t), \tag{1.1}$$

and

$$X^{(5)} + B_1 X^{(4)} + B_2 \dddot{X} + \Phi(\ddot{X}) + B_4 \dot{X} + H(X) = P_2(t), \tag{1.2}$$

in the real Euclidean space $R^n$ (with the usual norm denoted in what follows by $\|\cdot\|$) where $A_1, A_3, B_1, B_2, B_4$ are constant $n \times n$- matrices; $F, G, \Phi, H \in C^1[R^n,R^n]$ and $P_1, P_2 \in C^0[R,R^n]$. The matrices $A_1, A_3, B_1, B_2$ and $B_4$ that appeared in (1.1) and (1.2) are symmetric and the functions $P_1, P_2$ are both $\omega$-periodic in $t$, that is $P_i(t + \omega) = P_i(t), \ (i = 1, 2)$, for some $\omega > 0$ and all $t > 0, t \in R$. Let $J_f(\dot{X}), J_g(X), J_\phi(\ddot{X}), J_h(X)$ denote the

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Jacobian matrices corresponding to the functions $F(\tilde{X}), G(X), \Phi(\tilde{X}), H(X)$ respectively, that is

$$J_f(\tilde{X}) = \left(\frac{\partial f_i}{\partial x_j}\right), J_g(X) = \left(\frac{\partial g_i}{\partial x_j}\right), J_\phi(\tilde{X}) = \left(\frac{\partial \phi_i}{\partial x_j}\right), J_h(X) = \left(\frac{\partial h_i}{\partial x_j}\right)$$

where $(x_1, x_2, \ldots, x_n), (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n), (f_1, f_2, \ldots, f_n), (g_1, g_2, \ldots, g_n), (\phi_1, \phi_2, \ldots, \phi_n)$ and $(h_1, h_2, \ldots, h_n)$ are the components of $X, \tilde{X}, F, G, \Phi$ and $H$, respectively. It will be further assumed as basic throughout the paper that $J_f(\tilde{X}), J_g(X), J_\phi(\tilde{X}), J_h(X)$ are symmetric (for arbitrary $X \in \mathbb{R}^n$), so that their eigenvalues, which we denote respectively by $\lambda_i(J_g(X)), \lambda_i(J_h(X))$, $(i = 1, 2, \ldots, n)$, are all real.


$$X^{(4)} + A_1 \ddot{X} + A_2 \dot{X} + A_3 X + G(X) = P_1(t)$$

and

$$X^{(5)} + B_1 X^{(4)} + B_2 \ddot{X} + B_3 \dot{X} + B_4 X + H(X) = P_2(t).$$

According to the our observations in the relevant literature, we did not find another research with respect to the continuation of results established by Ezeilo [5]. It should be noticed that our results extend that obtained in [5]. However, till now, in a sequence of the works periodic properties for various third-, fourth-, fifth-, sixth-, seventh and eighth order certain nonlinear differential equations have been the subject of many investigations. (See, for example, Ezeilo ([4], [5]), Tejumola [9], Tunç ([13], [14], [15], [16]), and the references cited therein.)

We establish the following results.

**Theorem 1.** In addition to the fundamental assumptions imposed $F$ and $G$ in (1.1), suppose that following condition are satisfied:

Let $\delta_0 = \max_{i,j} \left| \frac{\partial f_i}{\partial x_j} \right|$ where $J_f(\tilde{X}) = \left(\frac{\partial f_i}{\partial x_j}\right)$, and suppose that there exists a constant $\alpha_1 > \frac{1}{4} n^2 \delta_0^2$ such that

$$\lambda_i(J_g(X)) \geq \alpha_1 \text{ for } i = 1, 2, \ldots, n \text{ and for arbitrary } X \in \mathbb{R}^n. \quad (1.3)$$

Then there exists at most one $\omega$-periodic solution of (1.1).
Theorem 2. Assume that $B_1$ is definite (positive or negative) and let

$$\beta_1 = \inf_i \lambda_i(B_1) \quad \text{or} \quad -\sup_i \lambda_i(B_1),$$

according as $B_1$ is positive or negative definite, where $\lambda_i(B_1) \ (i = 1, \ldots, n)$ are the eigenvalues of $B_1$. Let

$$\gamma_0 = \max_{i,j} \left| \frac{\partial \phi_i}{\partial x_j} \right|, \quad \text{where} \quad J_0(\ddot{X}) = \left( \frac{\partial \phi_i}{\partial x_j} \right).$$

Suppose that there exists a constant $\beta_2 > \frac{1}{4} n^2 \gamma_0^2 B_1^{-1}$ such that

$$k_1 \lambda_i(J_0(X)) \geq \beta_2 \quad (1.4)$$

where

$$k_1 = \begin{cases} 
+1, & \text{if } B_1 \text{ is positive definite} \\
-1, & \text{if } B_1 \text{ is negative definite.}
\end{cases}$$

Then there exists at most one $\omega$-periodic solution of (1.2).

We need the following algebraic result

Lemma. Let $A$ be a real symmetric $n \times n$ matrix and

$$a' \geq \lambda_i(A) \geq a > 0 \ (i = 1, 2, \ldots, n), \quad \text{where} \quad a', \ a \text{ are constants.}$$

Then

$$a' \langle X, X \rangle \geq \langle AX, X \rangle \geq a \langle X, X \rangle$$

and

$$a'^2 \langle X, X \rangle \geq \langle AX, AX \rangle \geq a^2 \langle X, X \rangle.$$

Proof. See [17]. \qed

2. Proof of the Theorem 1

Let $X_1(t), X_2(t)$ be any two solutions of (1.1) and set

$$Y(t) = X_2(t) - X_1(t).$$
Then $Y = Y(t)$ satisfies the differential equation

$$\dot{Y}^{(4)} + A_1 \ddot{Y} + S(t) \dot{Y} + A_3 \dot{Y} + R(t)Y = 0 \quad (2.1)$$

where the matrices $R(t)$ and $S(t)$ here are defined by

$$R(t) = \int_0^1 J_g(X_1(t) + \sigma(X_2(t) - X_1(t)))d\sigma, \quad (2.2)$$

$$S(t) = \int_0^1 J_f(\ddot{X}_1(t) + \sigma(\ddot{X}_2(t) - \ddot{X}_1(t)))d\sigma, \quad (2.3)$$

respectively. If $\langle \cdot, \cdot \rangle$, here and in what follows, denotes the usual scalar product in $\mathbb{R}^n$, that is $\langle U, V \rangle = \sum_{i=1}^n u_i v_i$ where $(u_1, u_2, \ldots, u_n)$, $(v_1, v_2, \ldots, v_n)$ are the respective components of $U, V \in \mathbb{R}^n$, it is clear, from the fact of $J_g(X), J_f(\ddot{X})$ being symmetric for all $X, \ddot{X}$, that $R(t), S(t)$ are symmetric and then from conditions of theorem that

$$\langle R(t)U, U \rangle \geq \alpha_1 \|U\|^2 \quad (2.4)$$

and

$$\langle S(t)V, W \rangle \geq -\delta_0 n\|V\|\|W\| \quad (2.5)$$

for all $t$ and for arbitrary $U, V, W \in \mathbb{R}^n$, respectively.

We shall now prove that, subject to (2.4) and (2.5), the equation (2.1) has no nontrivial $\omega$-periodic solutions, which will thereby verify the theorem.

Let then $Y = Y(t)$ be an $\omega$-periodic solution of (2.1) and consider the scalar function $\theta = \theta(t)$ defined by

$$\theta = \langle \dot{Y}, \ddot{Y} \rangle - \langle Y, \dot{Y} \rangle - \langle Y, A_1 \dot{Y} \rangle - \frac{1}{2} \langle Y, A_3 \dot{Y} \rangle + \frac{1}{2} \langle \dot{Y}, A_1 \dot{Y} \rangle.$$

We have, by an elementary differentiation, that

$$\dot{\theta} = \|\dot{Y}\|^2 + \langle S(t)Y, \ddot{Y} \rangle + \langle R(t)Y, Y \rangle$$

thus

$$\dot{\theta} \geq \|\dot{Y}\|^2 + \alpha_1 \|Y\|^2 - \delta_0 n\|Y\|\|\dot{Y}\|$$

$$= \left(\|\dot{Y}\| - \frac{1}{2} \delta_0 n\|Y\|\right)^2 + \left(\alpha_1 - \frac{1}{4} \delta_0^2 n^2\right)\|Y\|^2 \geq 0, \quad (2.6)$$
since
\[ \alpha_1 > \frac{1}{4} \frac{n^2 \delta_0^2}{q} . \]
Thus \( \theta(t) \) is nondecreasing in \( t \), and, being bounded (in view of the continuity and the assumed \( \omega \)-periodicity of \( Y(t) \)), it therefore tends to a unique limit as \( t \to \infty \). In particular, since
\[ \theta(t) = \theta(t + N\omega) \] (2.7)
for arbitrary \( t \) and for any integer \( N \), it follows then on letting \( N \to \infty \) in (2.7) that \( \theta(t) = \) constant, and therefore that
\[ \dot{\theta} (t) = 0 \] (2.8)
for all \( t \). It is clear from (2.6) and (2.8) that
\[ Y(t) \equiv 0 \] for all \( t \)
and the theorem now follows. \( \square \)

3. Proof of the Theorem 2

The procedure here is similar to that used above in section 2. If \( X_1(t) \), \( X_2(t) \) are any two solutions of (1.2), then \( Y = Y(t) = X_2(t) - X_1(t) \) satisfies the equation
\[ Y^{(5)} + B_1 Y^{(4)} + B_2 \dddot{Y} + M(t) \dddot{Y} + B_4 \dddot{Y} + N(t)Y = 0 \] (3.1)
where \( N(t) \) and \( M(t) \) are the symmetric matrices defined by
\[ N(t) = \int_0^1 J_h \left( X_1(t) + \sigma(X_2(t) - X_1(t)) \right) d\sigma \] (3.2)
and
\[ M(t) = \int_0^1 J_\phi \left( X_1(t) + \sigma(X_2(t) - X_1(t)) \right) d\sigma, \] (3.3)
respectively.
If (1.4) holds, then
\[ k_1 \langle N(t)U, U \rangle \geq \beta_2 \|U\|^2 \text{ for all } t \text{ and for arbitrary } U \in \mathbb{R}^n; \quad (3.4) \]
and the objective once again will be to show that, subject to (3.4), there are no nontrivial \(\omega\)-periodic solutions whatever of (3.1).

Let then \(Y = Y(t)\) be any \(\omega\)-periodic solution of (3.1) and consider the scalar function \(\psi = \psi(t)\) defined by
\[
\psi = \langle \ddot{Y}, \dot{Y} \rangle + \langle Y, B_1 \ddot{Y} \rangle - \langle Y, \dddot{Y} + B_1 \ddot{Y} + B_2 \dot{Y} \rangle \\
+ \frac{1}{2} \langle B_2 \ddot{Y}, \dot{Y} \rangle - \frac{1}{2} \langle \dddot{Y}, \dot{Y} \rangle - \frac{1}{2} \langle B_4 Y, Y \rangle.
\]
It is a straightforward matter to verify that
\[
\dot{\psi} = \langle B_1 \ddot{Y}, \dot{Y} \rangle + \langle N(t)Y, Y \rangle + \langle M(t) \ddot{Y}, Y \rangle,
\]
so that, by (3.3) and the definition of \(\gamma_0\),
\[
\dot{\psi} \geq \beta_1 \| \ddot{Y} \|^2 + \beta_2 \|Y\|^2 - \gamma_0 \| \dddot{Y} \| \| Y \|
\]
\[
= \beta_1 \left( \| \ddot{Y} \| - \frac{1}{2} n \gamma_0 \beta_1^{-1} \| Y \| \right)^2 + \left( \beta_2 - \frac{1}{4} n^2 \gamma_0^2 \beta_1^{-1} \right) \| Y \|^2 \quad (3.5)
\]
if \(B_1\) is positive definite, and
\[
\dot{\psi} \leq -\beta_1 \| \ddot{Y} \|^2 - \beta_2 \|Y\|^2 + \gamma_0 n \| \dddot{Y} \| \| Y \|
\]
\[
= -\beta_1 \left( \| \ddot{Y} \| - \frac{1}{2} n \gamma_0 \beta_1^{-1} \| Y \| \right)^2 - \left( \beta_2 - \frac{1}{4} n^2 \gamma_0^2 \beta_1^{-1} \right) \| Y \|^2 \quad (3.6)
\]
if \(B_1\) is negative definite. Thus, since \(\beta_2 > \frac{1}{4} n^2 \gamma_0^2 \beta_1^{-1}\), \(\psi(t)\) is monotone (increasing or decreasing according as \(B_1\) is positive or negative definite) in \(t\), and, being bounded, thus tends to a limit as \(t \to \infty\). As before this implies that \(\psi(t) = \text{constant}\) for all \(t\), and in turn, therefore, that
\[
\dot{\psi}(t) = 0 \text{ for all } t. \quad (3.7)
\]
It is evident from (3.5)-(3.7) that \(Y(t) = 0\) for all \(t\), and the theorem follows. \(\square\)
Remark. In the special case when the matrix $J_f(\dot{X}) = (\frac{\partial f}{\partial x_j})$ is diagonal, the estimate (2.6) can be readily refined to

$$\theta \geq \left( \| \dot{Y} \| - \frac{1}{2} \delta_0 \| Y \| \right)^2 + \left( \alpha_1 - \frac{1}{4} \delta_0^2 \right) \| Y \|^2$$

so that Theorem 1 holds here subject to the weaker condition $\alpha_1 > \frac{1}{4} \delta_0^2$ on $G$.

Similarly if the matrix $J_\phi(\dot{X}) = (\frac{\partial \phi}{\partial x_j})$ is diagonal, the estimates (3.5) and (3.6) can be relaxed respectively to

$$\psi \geq \beta_1 \left( \| \dot{Y} \| - \frac{1}{2} \gamma_0 \beta_1^{-1} \| Y \| \right)^2 + \left( \beta_2 - \frac{1}{4} \gamma_0^2 \beta_1^{-1} \right) \| Y \|^2,$$

$$\dot{\psi} \leq -\beta_1 \left( \| \dot{Y} \| - \frac{1}{2} \gamma_0 \beta_1^{-1} \| Y \| \right)^2 - \left( \beta_2 - \frac{1}{4} \gamma_0^2 \beta_1^{-1} \right) \| Y \|^2$$

so that Theorem 2 in this case holds subject to the weaker condition $\beta_2 > \frac{1}{4} \gamma_0^2 \beta_1^{-1}$.

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Faculty of Arts and Sciences, Department of Mathematics, Gaziosmanpaşa University, 60250, Tokat, TURKEY.

E-mail: ercantunc72@yahoo.com